Kalai - Smorodinsky solution to bargaining problems

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Nash bargaining problems

\((U, u) \in 2^\mathbb{R}^n \times \mathbb{R}^n\) is a Nash bargaining problem iff

- \(U\) is nonempty, compact and convex, and
- there exists \(u \in U\) such that \(u < u\)

let \(\mathcal{B}\) be the set of all bargaining problems
bargaining solutions

\[ s \in (\mathbb{R}^n)^B \] is a solution for Nash bargaining problems

iff

for all \((U, u) \in B\),

\[ s(U, u) \in U \]
desirable properties for bargaining solutions

(INV) **Invariance** to coordinate-wise affine transformations

A bargaining solution \( s \in (\mathbb{R}^n)^B \) is invariant to coordinate-wise affine transformations iff for all \((U, u), (U', u') \in B\) such that

1. there exist \(a \in \mathbb{R}_{++}^n\) and \(b \in \mathbb{R}^n\) such that

   \[
   \text{diag}(a)u + b = u'
   \]

2. and \(u \in U\) iff

   \[
   \text{diag}(a)u + b \in U'
   \]

it holds

\[
\text{diag}(a)s(U, u) + b = s(U', u')
\]
(SYM) **Symmetry** preservation

A bargaining solution \( s \in (\mathbb{R}^n)^B \) preserves symmetry iff

for all \((U, u)\) such that

1. for all \(i \neq j\), \(u_i = u_j\) and
2. for all \((u_1, \ldots, u_n) \in U\) and all bijective \(\rho \in \{1, \ldots, n\}\{1,\ldots,n\}\),

\[
(u_{\rho(1)}, \ldots, u_{\rho(n)}) \in U
\]

it holds, for all \(i \neq j\),

\[
s_i(U, u) = s_j(U, u)
\]
desirable properties for bargaining solutions

(EFF) **Efficiency**

A bargaining solution $s \in (\mathbb{R}^n)^B$ is efficient if and only if for all $(U, u) \in B$ and all $u \in U$ such that

1. $u \leq u$, and
2. there exists $u' \in U$ such that $u < u'$,

it holds

$$s(U, u) \neq u$$
(IND) **Independence** of irrelevant alternatives

A bargaining solution \( s \in (\mathbb{R}^n)^B \) is independent of irrelevant alternatives iff

for all \((U, u), (U', u') \in B\) such that

1. \( u = u' \),
2. \( U \subset U' \), and
3. \( s(U', u') \in U \),

it holds

\[
s(U, u) = s(U', u')
\]
(IND) **Independence** of irrelevant alternatives

A bargaining solution \( s \in (\mathbb{R}^n)^B \) is independent of irrelevant alternatives iff

for all \((U, u), (U', u') \in B\) such that

1. \( u = u' \),
2. \( U \subset U' \), and
3. \( s(U', u') \in U \),

it holds

\[
s(U, u) = s(U', u')
\]
(MON) **Monotonicity**

A bargaining solution $s \in (\mathbb{R}^n)^B$ is monotone iff

for all $(U, u), (U', u') \in B$ such that

1. $u = u'$,
2. $U \subset U'$, and
3. for some $i = 1, \ldots, n$, $\max \pi_i(U) = \max \pi_i(U')$,

it holds

$$s(U, u) = s(U', u')$$
(MON) **Monotonicity**

A bargaining solution \( s \in (\mathbb{R}^n)^B \) is monotone iff

For all \((U, u), (U', u') \in B\) such that

1. \( u = u' \),
2. \( U \subset U' \), and
3. For some \( i = 1, \ldots, n \), \( \max \pi_i(U) = \max \pi_i(U') \),

it holds that, for all \( j \neq i \)

\[
s_j(U, u) \leq s_j(U', u')
\]
Kalai-Smorodinsky theorem

$\hat{u}_1$, $\hat{u}_2$, $s(U, u)$, $U$
Kalai-Smorodinsky theorem

The Kalai-Smorodinsky solution to bargaining problems

\[
\text{s}(U, u)
\]

\[
U
\]

\[
U'
\]

\[
\hat{u}_1, \hat{u}_2
\]

\[
\hat{u}_1', \hat{u}_2'
\]
Kalai-Smorodinsky theorem

If \( s \in (\mathbb{R}^n)^B \) is

\[
\text{arg max}_{u \leq \hat{u} \in U \prod_{i=1}^n (u_i - \hat{u}_i)} u - \hat{u} = \lambda (\hat{u} - u), \text{ for some } \lambda > 0
\]
Kalai-Smorodinsky theorem

If \( s \in (\mathbb{R}^n)^B \) is

1. invariant to coordinate-wise affine transformations

\[
\text{arg max}_{u \leq \hat{u} \in U} \prod_{i=1}^n (u_i - \hat{u}_i) = \lambda (\hat{u} - u), \text{ for some } \lambda > 0
\]
Kalai-Smorodinsky theorem

If \( s \in (\mathbb{R}^n)^B \) is

1. invariant to coordinate-wise affine transformations
2. symmetry-preserving
Kalai-Smorodinsky theorem

If $s \in (\mathbb{R}^n)^B$ is

1. invariant to coordinate-wise affine transformations
2. symmetry-preserving
3. efficient
Kalai-Smorodinsky theorem

If $s \in (\mathbb{R}^n)^B$ is

1. invariant to coordinate-wise affine transformations
2. symmetry-preserving
3. efficient
4. monotone
Kalai-Smorodinsky theorem

If \( s \in (\mathbb{R}^n)^B \) is

1. invariant to coordinate-wise affine transformations
2. symmetry-preserving
3. efficient
4. monotone

then, for all \((U, u) \in B\),

\[
s(U, u) = \arg \max_{u \leq \hat{u} \in U} \prod_{i=1}^{n} (u_i - \hat{u}_i)
\]
Kalai-Smorodinsky theorem

If $s \in (\mathbb{R}^n)^B$ is

1. invariant to coordinate-wise affine transformations
2. symmetry-preserving
3. efficient
4. monotone

then, for all $(U, u) \in B$,

$$s(U, u) = \arg \max_{u \leq \hat{u} \in U} \prod_{i=1}^{n} (u_i - \hat{u}_i)$$

$$u - \hat{u} = \lambda (\hat{u} - u),$$ for some $\lambda > 0$
Kalai-Smorodinsky theorem

only $s$ satisfies INV, SYM, EFF, and MON:
only $s$ satisfies INV, SYM, EFF, and MON:

let $s \in (\mathbb{R}^n)^B$ satisfy INV, SYM, EFF, and MON and
only \( s \) satisfies INV, SYM, EFF, and MON:

let \( s \in (\mathbb{R}^n)^B \) satisfy INV, SYM, EFF, and MON and let \((U, u) \in B\)
only $s$ satisfies INV, SYM, EFF, and MON:

let $s \in (\mathbb{R}^n)^B$ satisfy INV, SYM, EFF, and MON and let $(U, u) \in B$

since

(i) $u < u$ for some $u \in U$, and (ii) for all $i$

$$\hat{u}^i = \max_{u \in U} u_i$$
only \( s \) satisfies INV, SYM, EFF, and MON:

let \( s \in (\mathbb{R}^n)^B \) satisfy INV, SYM, EFF, and MON and
let \( (U, u) \in B \)

since

(i) \( u < u \) for some \( u \in U \), and (ii) for all \( i \)

\[
\hat{u}^i = \max_{u \in U} u_i
\]

it follows that

\[
0 < \hat{u} - u
\]
Kalai-Smorodinsky theorem

The Kalai-Smorodinsky solution to bargaining problems
Kalai-Smorodinsky theorem

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Kalai-Smorodinsky solution to bargaining problems
only $s$ satisfies INV, SYM, EFF, and IND:

consider instead $(U', u')$ such that

$$\text{diag}(a)u + b = u'$$

and $u \in U$ iff

$$\text{diag}(a)u + b \in U'$$
only $s$ satisfies INV, SYM, EFF, and IND:

consider instead $(U', u')$ such that

$$\text{diag}(a)u + b = u'$$

and $u \in U$ iff

$$\text{diag}(a)u + b \in U'$$

where

$$a = + \left(\text{diag}(\hat{u} - u)\right)^{-1}$$

$$b = - \left(\text{diag}(\hat{u} - u)\right)^{-1} u$$
only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV

$$\text{diag}(a)s(U, u) + b = s(U', u')$$
only $s$ satisfies INV, SYM, EFF, and IND:

1 

by INV

$$\text{diag}(a)s(U,u) + b = s(U',u')$$

2 also

$$u^* = \arg \max_{u \leq u' \in U} \prod_{i=1}^{n} (u_i - u_i)$$

$$u - u = \lambda (\hat{u} - u), \text{ for some } \lambda > 0$$
only \( s \) satisfies INV, SYM, EFF, and IND:

1. by INV

\[
\text{diag}(a)s(U, u) + b = s(U', u')
\]

2. also, for all \( u \in U \) such that \( u \leq u \) and \( u - u = \lambda(\hat{u} - u) \), for some \( \lambda > 0 \),

\[
\prod_{i=1}^{n}(u_i - u_i) \leq \prod_{i=1}^{n}(u_i^{*} - u_i)
\]
only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV

$$\text{diag}(a)s(U, u) + b = s(U', u')$$

2. also, for all $u \in U$ such that $\underline{u} \leq u$ and $u - \underline{u} = \lambda(\hat{u} - u)$, for some $\lambda > 0$

$$\prod_{i=1}^{n}(a_iu_i + b_i - a_i\underline{u}_i - b_i) \leq \prod_{i=1}^{n}(a_iu_i^* + b_i - a_i\underline{u}_i - b_i)$$
only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV
   \[ \text{diag}(a)s(U, u) + b = s(U', u') \]

2. also, for all $u' \in U'$ such that $u' \leq u'$ and $u' - u' = \lambda(\hat{u}' - u')$, for some $\lambda > 0$
   \[ \prod_{i=1}^{n}(u'_i - u'_i) \leq \prod_{i=1}^{n}(a_i u^*_i + b_i - u'_i) \]
Kalai-Smorodinsky theorem

only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV

$$\text{diag}(a)s(U, u) + b = s(U', u')$$

2. also

$$\text{diag}(a)u^* + b = \arg \max_{u' \leq u' \in U'} \prod_{i=1}^{n} (u'_i - u'_i)$$

$$u' - u' = \lambda(\hat{u}' - u'), \text{ for some } \lambda > 0$$
Kalai-Smorodinsky theorem

only \( s \) satisfies INV, SYM, EFF, and IND:

1 by INV

\[
\text{diag}(a)s(\mathcal{U}, u) + b = s(\mathcal{U}', u')
\]

2 also

\[
\text{diag}(a)u^* + b = \arg \max_{u' \leq u' \in \mathcal{U}'} \prod_{i=1}^{n}(u'_i - u'_i) \\
\quad u' - u' = \lambda(\hat{u'} - u'), \text{ for some } \lambda > 0
\]

3 thus \( s(\mathcal{U}, u) = u^* \) if, and only if,

\[
s(\mathcal{U}', u') = \arg \max_{u' \leq u' \in \mathcal{U}'} \prod_{i=1}^{n}(u'_i - u'_i) \\
\quad u' - u' = \lambda(\hat{u'} - u'), \text{ for some } \lambda > 0
\]
Kalai-Smorodinsky theorem

only \( s \) satisfies INV, SYM, EFF, and IND:

1 by INV

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\text{diag}(a)s(U, u) + b = s(U', u')
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\text{diag}(a)u^* + b = \arg \max_{u' \leq u' \in U'} \prod_{i=1}^{n}(u'_i - u'_i)
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u' - u' = \lambda(\hat{u}' - u'), \text{ for some } \lambda > 0
\]

3 thus \( s(U, u) = u^* \) if, and only if,

\[
s(U', u') = \arg \max_{u' \leq u' \in U'} \prod_{i=1}^{n}(u'_i - u'_i)
\]

\[
u' - u' = \lambda(\hat{u}' - u'), \text{ for some } \lambda > 0
\]
only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV
   \[ \text{diag}(a)s(U, u) + b = s(U', u') \]

2. also
   \[
   \text{diag}(a)u^* + b = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i \\
   u' = \lambda \hat{u}', \text{ for some } \lambda > 0
   \]

3. thus $s(U, u) = u^*$ if, and only if,
   \[
   s(U', 0) = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i \\
   u' = \lambda \hat{u}', \text{ for some } \lambda > 0
   \]
only $s$ satisfies INV, SYM, EFF, and IND:

1. by INV
   \[
   \text{diag}(a)s(U, u) + b = s(U, u)
   \]

2. also
   \[
   \text{diag}(a)u^* + b = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i
   \]
   \[
   u' = \lambda \hat{u}', \text{ for some } \lambda > 0
   \]

3. thus $s(U, u) = u^*$ if, and only if,
   \[
   s(U', 0) = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i
   \]
   \[
   u' = \lambda \hat{u}', \text{ for some } \lambda > 0
   \]
Kalai-Smorodinsky theorem

only $s$ satisfies INV, SYM, EFF, and IND:

1 by INV

$$\text{diag}(a)s(U, u) + b = s(U', u')$$

2 also

$$\text{diag}(a)u^* + b = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i$$

$$u' = \lambda \mathbf{1}, \text{ for some } \lambda > 0$$

3 thus $s(U, u) = u^*$ if, and only if,

$$s(U', 0) = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i$$

$$u' = \lambda \mathbf{1}, \text{ for some } \lambda > 0$$
Kalai-Smorodinsky theorem

Kalai-Smorodinsky solution to bargaining problems
Let
\[ \tilde{U}' \] be the comprehensive hull of \( U' \)
Let

$\tilde{U}'$ be the comprehensive hull of $U'$

$U''$ be the convex hull of $\text{diag}(a)u^* + b$ and $\{e_i\}_{i=1}^n$
Kalai-Smorodinsky theorem
Kalai-Smorodinsky theorem

$\tilde{U} \cup \tilde{U}'$
Kalai-Smorodinsky theorem

\[ \begin{align*}
\hat{u}_1 & \quad \hat{u}_2 \\
\tilde{U} & \quad U
\end{align*} \]

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Kalai - Smorodinsky solution to bargaining problems
Kalai-Smorodinski Theorem

1 \( s(U', 0) = s(\tilde{U}', 0) \)
Kalai-Smorodinski Theorem

1\hspace{10pt} s(U', 0) = s(\tilde{U}', 0)

since

(1) \hspace{10pt} U' \subset \tilde{U}'
(2) \hspace{10pt} \max_{u \in U'} u_i = \max_{u \in \tilde{U}'} u_i, \text{ for all } i
(3) \hspace{10pt} s \text{ is MON}

\hspace{40pt} s(U', 0) \leq s(\tilde{U}', 0)
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
   
   since
   
   (1) \( U' \subseteq \tilde{U}' \)
   
   (2) \( \max_{u \in U'} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \)
   
   (3) \( s \) is MON

   \[ s(U', 0) \leq s(\tilde{U}', 0) \]

   since also
   
   \( s(\tilde{U}', 0) \in P_{\tilde{U}'} = P_{U'} \subseteq U' \)

   \( s \) is EFF

   \( s(U', 0) \leq s(\tilde{U}', 0) \)

   \[ s(U', 0) = s(\tilde{U}', 0) \]
Kalai-Smorodinski Theorem

1 \( s(U', 0) = s(\tilde{U}', 0) \)
2 \( s(U'', 0) = s(\tilde{U}', 0) \)

since

1 \( U'' \subset \tilde{U}' \),
2 \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
3 \( s \) is MON,

\( s(U'', 0) \leq s(\tilde{U}', 0) \) since also

1 \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal),
2 \( s \) is EFF and SYM.

\[ S(U'', 0) = \arg \max_{0 \leq u \in U''} n \prod_{i=1} u_i = \lambda_1, \text{ for some } \lambda > 0 \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since
(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[ s(U'', 0) \leq s(\tilde{U}', 0) \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since

(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[ s(U'', 0) \leq s(\tilde{U}', 0) \]

since also

(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[ S(U'', 0) = \arg \max_{0 \leq u \in U''} \prod_{i=1}^{n} u_i; \]

\[ u = \lambda 1, \text{ for some } \lambda > 0 \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since
(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[ s(U'', 0) \leq s(\tilde{U}', 0) \]

since also
(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[ S(U'', 0) = \text{diag}(a)u^* + b \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since

(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[
s(U'', 0) \leq s(\tilde{U}', 0)
\]

since also

(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[
S(U'', 0) = \text{diag}(a)u^* + b \in P_{\tilde{U}'}
\]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since

(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[ s(U'', 0) = s(\tilde{U}', 0) \]

since also

(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[ S(U'', 0) = \text{diag}(a)u^* + b \in P_{\tilde{U}'} \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since
(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[ s(U'', 0) = s(U', 0) \]

since also
(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[ S(U'', 0) = \text{diag}(a)u^* + b \in P_{\tilde{U}'} \]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since

(1) \( U'' \subset \tilde{U}' \),
(2) \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
(3) \( s \) is MON,

\[
s(U', 0) = \text{diag}(a)u^* + b
\]

since also

(1) \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
(2) \( s \) is EFF and SYM

\[
S(U'', 0) = \text{diag}(a)u^* + b \in P_{\tilde{U}'}
\]
Kalai-Smorodinski Theorem

1. \( s(U', 0) = s(\tilde{U}', 0) \)
2. \( s(U'', 0) = s(\tilde{U}', 0) \)

since

1. \( U'' \subset \tilde{U}' \),
2. \( \max_{u \in U''} u_i = \max_{u \in \tilde{U}'} u_i \), for all \( i \),
3. \( s \) is MON,

\[
s(U', 0) = \arg \max_{0 \leq u' \in U'} \prod_{i=1}^{n} u'_i
\]

\( u' = \lambda 1 \), for some \( \lambda > 0 \)

since also

1. \( U'' \) is symmetric (\( \text{diag}(a)u^* + b \) is on the diagonal)
2. \( s \) is EFF and SYM

\[
S(U'', 0) = \text{diag}(a)u^* + b \in P_{\tilde{U}'}
\]
Consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]
consider the economy

\[
\begin{align*}
    u_1(x, y) &= xy & \text{and } (\bar{x}_1, \bar{y}_1) &= (1, 0) \\
    u_2(x, y) &= x^2 y & \text{and } (\bar{x}_2, \bar{y}_2) &= (0, 1)
\end{align*}
\]

Walrasian allocation?
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Walrasian allocation?

\[ y_1(x_1 - 1) + x_1(y_1 - 0) = 0 \]
\[ 2x_2y_2(x_2 - 0) + x_2^2(y_2 - 1) = 0 \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Walrasian allocation?

\[
\begin{align*}
y_1(x_1 - 1) + x_1(y_1 - 0) &= 0 \\
2x_2y_2(x_2 - 0) + x_2^2(y_2 - 1) &= 0
\end{align*}
\]

\[
\begin{align*}
x_1 &= \frac{1}{2} \\
y_2 &= \frac{1}{3}
\end{align*}
\]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2 y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Walrasian allocation?

\[ y_1(x_1 - 1) + x_1(y_1 - 0) = 0 \]
\[ 2x_2y_2(x_2 - 0) + x_2^2(y_2 - 1) = 0 \]

\[ (x_1, y_1) = (\frac{1}{2}, \frac{2}{3}) \]
\[ (x_2, y_2) = (\frac{1}{2}, \frac{1}{3}) \]
Consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[ (x_1, y_1) = (?, ?) \]
\[ (x_2, y_2) = (?, ?) \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[
\max x_1y_1 \cdot x_2^2 y_2 \\
x_1 + x_2 = 1 \\
y_1 + y_2 = 1 \\
(x_1, y_1) = (?, ?) \\
(x_2, y_2) = (?, ?)
\]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[
\begin{pmatrix}
y_1 \cdot x_2^2 y_2 \\
x_1 \cdot x_2^2 y_2 \\
x_1 y_1 \cdot 2x_2y_2 \\
x_1 y_1 \cdot x_2^2
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix} + \mu
\begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix}
\]

\[ (x_1, y_1) = (?, ?) \]
\[ (x_2, y_2) = (?, ?) \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2 y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[ y_1 \cdot x_2^2 y_2 = x_1 y_1 \cdot 2x_2 y_2 \]
\[ x_1 \cdot x_2^2 y_2 = x_1 y_1 \cdot x_2^2 \]

\[ (x_1, y_1) = (?, ?) \]
\[ (x_2, y_2) = (?, ?) \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2 y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[ y_1 \cdot x_2^2 y_2 = x_1 y_1 \cdot 2 x_2 y_2 \]
\[ x_1 \cdot x_2^2 y_2 = x_1 y_1 \cdot x_2^2 \]

\( (x_1, y_1) = (?, ?) \)
\( (x_2, y_2) = (?, ?) \)
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[ x_2 = 2x_1 \]
\[ y_2 = y_1 \]

\((x_1, y_1) = (?, ?)\)
\((x_2, y_2) = (?, ?)\)
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2 y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[ x_2 = 2x_1 \]
\[ y_2 = y_1 \]

\( (x_1, y_1) = (\frac{1}{3}, \frac{1}{2}) \)
\( (x_2, y_2) = (\frac{2}{3}, \frac{1}{2}) \)
consider the economy

\[ u_1(x, y) = xy \] and \( (\bar{x}_1, \bar{y}_1) = (1, 0) \)

\[ u_2(x, y) = x^2y \] and \( (\bar{x}_2, \bar{y}_2) = (0, 1) \)

Kalai-Smorodinsky allocation?

\[ (x_1, y_1) = (?, ?) \]

\[ (x_2, y_2) = (?, ?) \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Kalai-Smorodinsky allocation?

\[
\max x_1y_1 \cdot x_2^2y_2 \\
x_1 + x_2 = 1 \\
y_1 + y_2 = 1
\]

\((x_1, y_1) = (?, ?)\)
\((x_2, y_2) = (?, ?)\)
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Kalai-Smorodinsky allocation?

\[
\begin{align*}
\text{max } x_1 y_1 \cdot x_2^2 y_2 \\
x_1 + x_2 = 1 \\
y_1 + y_2 = 1 \\
x_2^2 y_2 - x_1 y_1 = 0
\end{align*}
\]

\[(x_1, y_1) = (?, ?)\]
\[(x_2, y_2) = (?, ?)\]
Consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[
\begin{pmatrix}
  y_1 \cdot x_2^2 y_2 \\
  x_1 \cdot x_2^2 y_2 \\
  x_1 y_1 \cdot 2x_2 y_2 \\
  x_1 y_1 \cdot x_2^2
\end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \nu \begin{pmatrix} -y_1 \\ -x_1 \\ 2x_2 y_2 \\ x_2^2 \end{pmatrix}
\]

\[ (x_1, y_1) = (?, ?) \]
\[ (x_2, y_2) = (?, ?) \]
consider the economy

\[ u_1(x, y) = xy \quad \text{and} \quad (\bar{x}_1, \bar{y}_1) = (1, 0) \]
\[ u_2(x, y) = x^2y \quad \text{and} \quad (\bar{x}_2, \bar{y}_2) = (0, 1) \]

Nash bargaining allocation?

\[
\frac{y_{12}(2x_1 - x_2)}{x_1x_2(y_1 - y_2)} = \frac{2x_2y_2 + y_1}{x_2^2 + x_1}
\]

\[ x_2^2 y_2 = x_1 y_1 \]
\[ x_1 + x_2 = 1 \]
\[ y_1 + y_2 = 1 \]

\[(x_1, y_1) = (.3611, .5306)\]
\[(x_2, y_2) = (.6389, .4694)\]
Kalai-Smorodinsky theorem

Julio Dávila

Kalai - Smorodinsky solution to bargaining problems