VALUATION EQUILIBRIUM AND PARETO OPTIMUM IN NON-CONVEX ECONOMIES*

Jean-Marc BONNISSEAU and Bernard CORNET

CORE, Université Catholique de Louvain, Louvain-le-Neuve, Belgium
CERMSEM, Université Paris 1 Panthéon-Sorbonne, Paris, France

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In this paper, we report an extension of the second welfare theorem when both convexity and differentiability assumptions are violated. Our model allows various formalizations of the marginal rule and considers the general setting of a topological vector space of commodities.

1. Introduction

The second welfare theorem for convex economies states that under appropriate assumptions, at a Pareto optimal allocation, it can be associated with a non-zero price vector such that each consumer minimizes his (her) expenditure and each firm maximizes its profit or, equivalently, by a redistribution of income, the optimal allocation can be obtained as a Walrasian equilibrium. This result was firstly rigorously formulated and proved by Arrow (1951) and Debreu (1951) in an economy with a finite number of commodities and by Debreu (1954) to allow infinitely many commodities. In the latter case, a topological vector space of commodities is considered and Debreu's result mainly rests on the assumption that the aggregate production set of the economy has a non-empty interior.

When the convexity assumptions are violated, in particular, in the presence of increasing returns, Guesnerie (1975) proposed to associate with a Pareto optimal allocation, a non-zero price vector such that each consumer will satisfy the 'first-order necessary conditions' for expenditure minimization and each firm will satisfy the 'first-order necessary conditions' for profit maximization. Under convexity assumptions on the preferences of the consumers and on the production sets of the firms, the 'necessary conditions' become also sufficient, so that the above statement coincides with the standard (convex) formulation of the second welfare theorem. Several formalizations and proofs of the above statement are now available for finite or infinite

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dimensional commodity spaces. Apart from their generality, these results differ in the way the 'necessary conditions' (or the marginal rule) are mathematically formalized in the absence of convexity and differentiability assumptions. For this formalization, Guesnerie uses the concept of tangent (normal) cone of Dubovickii and Miljutin (1965), whereas Kahn and Vohra (1984, 1985), Yun (1984), Quinzii (1986) and Cornet (1986) use the concept of tangent (normal) cone in the sense of Clarke (1975) which was first introduced by Cornet (1982) in the related problem of existence of equilibria in non-convex economies. At this stage, it was worth pointing out that Guesnerie's result and the other ones (which use a different formalization of the marginal rule) are not directly comparable, i.e., neither Guesnerie's implies one of the other results, nor the converse is true (see Remark 3.1). We point out that, of all the above papers, Kahn and Vohra (1985) is the only one to allow infinitely many commodities in the spirit of Debreu (1954) and that furthermore they also consider an economy with public goods.

We consider here the general setting of a topological vector space of commodities but we exclude the presence of public goods. Our first result (Theorem 2.1) provides a (non-convex) version of the second welfare theorem with the marginal rule formalized by Clarke's normal cone. This theorem generalizes the results of Debreu (1954) and Kahn and Vohra (1985) in the absence of public goods. In an economy with a finite number of commodities, Theorem 2.1 also extends the results of Kahn and Vohra (1984), Quinzii (1986) and Yun (1984). In our second result (Theorem 3.1), we consider different ways to formalize the 'first-order necessary conditions' (or the marginal rule), which include Clarke's one and also the one of Dubovickii and Miljutin used by Guesnerie. This will allow us to provide a generalization of Guesnerie's result (Corollary 3.1) in an infinite dimensional setting and also to show the relationship between this result and the above quoted ones.

The paper is organized as follows. In section 2, we recall definitions and properties of Clarke's tangent and normal cones which will be used later and we state our first result, Theorem 2.1. In section 3, we state our second result (Theorem 3.1) which implies the previous one and Guesnerie's. Also some additional definitions and properties of tangent and normal cones are recalled. In section 4, the proofs of these results are given.

2. The model and a first result

We consider an economy with $m$ consumers, $n$ firms and we assume that the commodity space $L$ is a real topological vector space. We let $X_i \subset L$ be the consumption set of the $i$th consumer and, for $x=(x_1,\ldots,x_m)$ in $X_1 \times \cdots \times X_m$, we let $P_i(x)$ be the set of elements in $X_i$ which are preferred to
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By the $i$th consumer. In other words, the preferences of the $i$th consumer are described by a correspondence $P_i$, from $X_1 \times \cdots \times X_m$ to $X_i$. We let $Y_i \subset L$ be the production set of the $j$th firm and $\omega$ in $L$ be the vector of total initial endowments.

Definition 2.1. An $(m+n)$-tuple $((x_i^*), (y_j^*))$ of elements of $L$ is said to be a feasible allocation if, (a) for all $i$, $x_i^* \in X_i$, (b) for all $j$, $y_j^* \in Y_j$ and (c) $\sum_{i=1}^m x_i^* - \sum_{j=1}^n y_j^* - \omega = 0$. An $(m+n)$-tuple $((x_i^*), (y_j^*))$ of elements of $L$ is said to be a Pareto optimum (resp. a weak Pareto optimum) if it is a feasible allocation and if there exists no feasible allocation $((x_i^*), (y_j^*))$ such that, for all $i$, $x_i \in \text{cl } P_i(x_1^*, \ldots, x_m^*)$, the closure of $P_i(x_1^*, \ldots, x_m^*)$, and, for some $i_0 \in \{1, \ldots, m\}$, $x_{i_0} \in P_{i_0}(x_1^*, \ldots, x_m^*)$ (resp., for all $i$, $x_i \in P_i(x_1^*, \ldots, x_m^*)$).

In the following, we shall simply denote $P_i(x_1^*, \ldots, x_m^*)$ by $P_i^*$.

Before stating our first result, we need to introduce some general definitions together with the concept of tangent and normal cones in the sense of Clarke. Firstly, we let $L^*$ be the set of all continuous real-valued linear functions defined on $L$, and $\mathcal{N}$ be the set of open neighborhoods of 0 in $L$. If $x \in L$ and $p \in L^*$ we denote $p(x)$ by $p \cdot x$ and, for $C \subset L$, we let $\text{cl } C$, $\text{int } C$, $C^0 = \{p \in L^* | p \cdot x \leq 0, \text{ for all } x \in C\}$ denote, respectively, the closure, the interior and the negative polar cone of $C$. Let $x$ be an element in $\text{cl } C$, we then define the set $H_C(x)$ of hypertangent vectors to $C$ at $x$, Clarke's tangent and normal cones, $T_C(x)$ and $N_C(x)$, to $C$ at $x$, as follows:

$$H_C(x) = \left\{ v \in L \mid \exists \varepsilon > 0, \exists N \in \mathcal{N}, \forall t \in (0, \varepsilon), \forall x' \in (x+N) \cap C : x' + t(v+N) \subset C \right\},$$

$$T_C(x) = \left\{ v \in L \mid \forall N \in \mathcal{N}, \exists \varepsilon > 0, \exists N' \in \mathcal{N}, \forall t \in (0, \varepsilon), \forall x' \in (x+N') \cap C : [x' + t(v+N)] \cap C \neq \emptyset \right\},$$

$$N_C(x) = T_C(x)^0.$$

The above definitions of $T_C(x)$ and $N_C(x)$, due to Rockafellar (1979, 1980), are extensions of the original ones of Clarke (1975) to the setting of topological vector spaces. It is worth noticing that an element $v$ in $L$ belongs to $T_C(x)$, if and only if, for all generalized sequences (nets) $\{x_k^*\} \subset C$, $\{t_k\} = (0, + \infty)$ converging, respectively, to $x$ and 0, there exists a generalized sequence $\{v_k^*\} \subset E$ converging to $v$ such that, for all $k$, $x_k^* + t_k v_k^* \in C$. Hence, when $L$ is a Banach space, the above definition coincides with the original one, by Clarke (1983, Theorem 2.4.5). The definition of $H_C(x)$ is taken from Clarke (1983) but it is different from the one in Rockafellar (1980) [who considers a set, the interior of which contains $H_C(x)$].
Definition 2.2. A subset $C$ of $L$ is said to be epi-Lipschitzian at $x$, an element in $\text{cl} C$, if there exists an hypertangent vector to $C$ at $x$, or equivalently, if $H_C(x)$ is non-empty. The set $C$ is said to be epi-Lipschitzian if it is epi-Lipschitzian at every element $x$ in $\text{cl} C$.

The following proposition gives two cases, of particular economic interest under which a set is epi-Lipschitzian. It summarizes also the properties of the above cones which will be used in the following. We recall that a subset $C$ of $L$ is said to be a cone (with vertex the origin) if $\lambda x$ belongs to $C$, for all $\lambda > 0$ and $x \in C$.

Proposition 2.1. Let $C$ be a subset of $L$ and let $x$ be an element in $\text{cl} C$, then the following assertions hold true.

(a) $T_{\text{cl}}(x) = T_C(x)$, $N_{\text{cl}}(x) = N_C(x)$ and $H_{\text{cl}}(x) \subseteq H_{\text{cl}}(x)$.

(b) $T_C(x)$ is a closed, convex cone, containing 0; $H_C(x)$ and $H_{\text{cl}}(x)$ are open convex cones (possibly empty) contained in $T_C(x)$.

(c) If $H_C(x)$ [resp. $H_{\text{cl}}(x)$] is non-empty, then $\text{int} T_C(x) = H_C(x)$ [resp. $\text{int} T_C(x) = H_{\text{cl}}(x)$].

(d) Let us assume that $C$ is convex. Then, $H_C(x) = \{\lambda (x' - x) | \lambda > 0, x' \in \text{int} C\}$, $T_C(x) = \text{cl} \{\lambda (x' - x) | \lambda > 0, x' \in C\}$ and $N_C(x) = \{p \in L^* | p \cdot x \leq p \cdot x', \text{for all } x' \in C\}$. Hence, $C$ is epi-Lipschitzian if and only if $C$ has a non-empty interior.

(e) Let us assume that $C + Q \subseteq C$, for some non-empty open cone $Q$. Then $Q \subseteq H_C(x)$ and $C$ is epi-Lipschitzian.

The proofs of (a), (b) and (e) are immediate, at the exception of the fundamental properties, in (b), that $T_C(x)$, $H_C(x)$ and $H_{\text{cl}}(x)$ are convex. When $L = \mathbb{R}^n$, the convexity of $T_C(x)$, as defined above, is proved by Rockafellar (1979) who then noticed that his argument actually carries to an arbitrary topological vector space $L$ [Rockafellar (1980, p. 258)]. The convexity of $H_C(x)$ and $H_{\text{cl}}(x)$ is a consequence of the convexity of $T_C(x)$ and of (c), which is exactly Corollary 2 of Rockafellar (1980). Assertion (d) is Theorem 1 of Rockafellar (1980) when $L$ is locally convex; a proof of it, together with a more general assertion (Proposition 3.1.d) will be given in section 4 when $L$ is an arbitrary topological vector space. Finally we point out that all the assertions of the proposition are proved in the book of Clarke (1983) when $L$ is a Banach space.

We now state the main result of this section.

Theorem 2.1. Let $(x^*_i,(y^*_i))$ be a weak Pareto optimum (resp. a Pareto optimum and assume $m > 1$) such that, for all $i$, $x_i^* \in \text{cl} P_i^*$ and, either, for some $i$,
$P_i^*$ (resp. $\text{cl } P_i^*$) is epi-Lipschitzian at $x_i^*$, or, for some $j$, $Y_j$ is epi-Lipschitzian at $y_j^*$.

Then, there exists a non-zero price vector $p^*$ in $L^*$ satisfying

(a) for all $i$, $-p^* \in N_{P_i^*}(x_i^*)$;
(b) for all $j$, $p^* \in N_{Y_j}(y_j^*)$.

The proof of Theorem 2.1 will be given in section 4 as a consequence of a more general result, Theorem 3.1, which will be stated in the next section. We notice that, if $P_i^*$ is epi-Lipschitzian at $x_i^*$, then $\text{cl } P_i^*$ is also epi-Lipschitzian at $x_i^*$, since $H_{P_i^*}(x_i^*) \subset H_{\text{cl } P_i^*}(x_i^*)$ [Proposition 2.1(a)], but the converse is not true, in general, even in the convex case. We also point out that, if $m=1$, the notions of Pareto optimality and of weak Pareto optimality are identical.

At this stage, it is worth discussing the link between Theorem 2.1 and previous work in the literature on the same subject. The above theorem generalizes a previous result of Kahn and Vohra (1985) who make the stronger assumptions that, for all $i$, $P_i^*$ is epi-Lipschitzian and, for all $j$, $Y_j$ is epi-Lipschitzian and that the commodity space $L$ is ordered and locally convex. When $L$ is finite dimensional, Theorem 2.1 can be generalized in several directions, for which we refer to Cornet (1986). However, Theorem 2.1 also generalizes the results of Kahn and Vohra (1984), Quinzii (1986) and Yun (1984) who all assume that $L=\mathbb{R}_+^l$, and that, for all $i$, $P_i^* + \mathbb{R}_+^l \subset P_i^*$ and, for all $j$, $Y_j - \mathbb{R}_+^l \subset Y_j$ [which, by Proposition 2.1(e), implies that the sets $P_i^*$ and $Y_j$ are epi-Lipschitzian]. It is worth also pointing out that our definition of Pareto optimality is slightly different from the one used in Kahn and Vohra (1984, 1985), Quinzii (1986) and Yun (1984), who assume implicitly a free-disposal assumption in their definition, i.e., the condition (c) in Definition 2.1 is replaced by (c'): $\sum_{i=1}^n x_i^* - \sum_{j=1}^m y_j^* - \omega \leq 0$, which requires that the space is ordered [see the discussion in Cornet (1986), which is also valid in the infinite dimensional setting]. The link with the result of Guesnerie (1975) will be discussed in the next section.

Remark 2.1. We now assume that $\mathcal{E}=(X_i, P_i, Y_j, \omega)$ is a convex economy, in the sense that, for all $i$, all $x$ in $\prod_i X_i$, $P_i(x)$ is convex and, for all $j$, $Y_j$ is convex. Then, by Proposition 2.1(d), the conditions (a) and (b) hold if and only if

(a') for all $i$, $x_i^*$ minimizes $p^* \cdot x_i$ over $\text{cl } P_i^*$, and
(b') for all $j$, $y_j^*$ maximizes $p^* \cdot y_j$ over the production set $Y_j$.

Hence, in the terminology of Arrow and Hahn (1971), $((x_i^*), (y_j^*))$ is a compensated equilibrium, also called price quasi-equilibrium by Mas-Colell (1985). For the (standard) way to go, from price quasi-equilibria to price
equilibria with respect to a price system [Debreu (1959)], i.e., from expenditure minimization to preference maximization by the consumers, we refer to the three above books.

From the above remark, Proposition 2.1(d) and Theorem 2.1 (applied to the economy $\mathcal{E}'=((X_i, P_i), \sum_{j=1}^n Y_j, \omega)$) one deduces the infinite dimensional version of the second welfare theorem for convex economies.

**Corollary 2.1.** [Debreu (1954)]. The Pareto optimum $((x^*_t), (y^*_t))$ is a price quasi-equilibrium if, for all $i$, $x^*_t \in cl P^*_t$, $P^*_t$ is convex, $\sum_{j=1}^n Y_j$ is convex and, either $\sum_{j=1}^n Y_j$ or $P^*_t$, for some $i$, has a non-empty interior.

**Remark 2.2.** A second case of particular economic interest, under which either $P^*_t$ or $Y_j$ may be epi-Lipschitzian is given by Proposition 2.1(e) and is pointed out by Kahn and Vohra (1985). Let us suppose that the commodity space $L$ is an ordered topological vector space whose positive cone $Q$ has a non-empty interior. Then $Y_j$ is epi-Lipschitzian if $Y_j - Q \subset Y_j$, which is the standard free-disposal assumption, and $P^*_t$ (resp. $cl P^*_t$) is epi-Lipschitzian at $x^*_t$ if $P^*_t + int Q \subset cl P^*_t$ (resp. $cl P^*_t + int Q \subset cl P^*_t$), which is satisfied if the preferences are monotonic and transitive. The assumption that the positive cone $Q$ has a non-empty interior is satisfied when (i) $L$ is finite dimensional or (ii) $L= L_\infty$ is the space of bounded functions on a measure space, endowed with the supremum norm, a commodity space first considered by Bewley (1972) in the related problem of existence of equilibria.

When the positive cone has an empty interior, as it is the case in several examples of ordered commodity spaces of economic importance [see Mas-Colell (1986)], the above discussion is no longer valid. One can expect, however, to have, in the non-convex case, results analogue to the ones of Mas-Colell (1986) in the convex case. In this paper we do not consider this question and we only provide non-convex analogues of Debreu's (1954) result.

3. A further result and the link with Guesnerie's

Before stating the main result of this section we first need to introduce some additional definitions of tangent and normal cones (which will be, respectively, larger and smaller than Clarke's cones). If $C$ is a subset of $L$ and $x$ is an element of $cl C$ we let

$$k_C(x) = \{ v \in L \mid \exists \varepsilon > 0, \exists N \in \mathcal{N}, \forall t \in (0, \varepsilon) : x + t(v + N) \subset C \},$$

$$t_C(x) = \{ v \in L \mid \forall N \in \mathcal{N}, \exists \varepsilon > 0, \forall t \in (0, \varepsilon) : [x + t(v + N)] \cap C \neq \emptyset \},$$
\( n_c(x) = t_c(x)^0. \)

The first set \( k_c(x) \) is the cone of interior displacement of Dubovickii and Miljutin (1965), \( t_c(x) \) is a related notion of tangent cone and \( n_c(x) \) is the negative polar cone of \( t_c(x) \). The above definitions are very much the analogue of the ones of \( H_c(x) \), \( T_c(x) \) and \( N_c(x) \) in section 2. There is, however, a major drawback, since the cones \( k_c(x) \) and \( t_c(x) \) are no longer convex, in general, and we have lost for these cones the fundamental convexity properties of the cones \( H_c(x) \) and \( T_c(x) \), which make them so easy to work with.

The next proposition summarizes the properties of these cones, which will be used in the following.

**Proposition 3.1.** Let \( C \) be a subset of \( L \) and let \( x \) be an element of \( \text{cl} \, C \), then the following assertions hold true.

(a) \( t_c(x) \) is a closed cone, containing 0; \( k_c(x) \) and \( k_{clc}(x) \) are open cones (possibly empty).

(b) \( H_c(x) \subseteq k_c(x) \subseteq k_{clc}(x) \subseteq \text{int} \, t_c(x); \) \( H_{clc}(x) \subseteq k_{clc}(x) \subseteq \text{int} \, t_c(x). \)

(c) \( T_c(x) \subseteq t_c(x) = t_{clc}(x) \subseteq \text{cl} \{ \lambda (x' - x) | \lambda > 0, x' \in C \}, \)
\( N_c(x) = n_c(x) = n_{clc}(x) = \{ p \in L^* | p \cdot x \leq p \cdot x', \text{ for all } x' \in C \}. \)

(d) Let us assume that \( C \) is convex. Then all the inclusions in (c) are equalities and \( H_c(x) = k_c(x) = \{ \lambda (x' - x) | \lambda > 0, x' \in \text{int} \, C \}. \) Furthermore, \( C \) is epi-Lipschitzian if and only if \( C \) has a non-empty interior.

The proofs of (a), (b) and (c) are straightforward. The proof of (d) [which implies assertion (d) of Proposition 2.1] will be given in section 4.

We can now state the main result of this section.

**Theorem 3.1.** Let \( ((x_i^*),(y_j^*)) \) be a weak Pareto optimum (resp. a Pareto optimum and assume that \( m > 1 \)) such that, for all \( i, x_i^* \in \text{cl} \, P_i^* \). Let \( T_i \) \( (i = 1, \ldots, m) \), \( T_j \) \( (j = 1, \ldots, n) \) be convex cones of \( L \) such that for all \( i, T_i \subseteq \text{int} \, P_i^* (x_i^*), \) for all \( j, T_j \subseteq \text{cl} \, Y_j(y_j^*), \) and
\[
\begin{cases}
\text{either, for some } i, \emptyset \neq \text{int} \, T_i \subseteq k_{P_i^*}(x_i^*) \ [\text{resp. } k_{clP_i^*}(x_i^*)], \\
or, \text{ for some } j, \emptyset \neq \text{int} \, T_j \subseteq k_{Y_j}(y_j^*).
\end{cases}
\]

Then, there exists a non-zero price vector \( p^* \) in \( L^* \) satisfying

\[
\begin{align*}
    \text{Nash equilibrium:} & \quad \sum_i p_i^* x_i^* = \sum_j p_j^* y_j^* \\
    \text{Pareto optimality:} & \quad p^* \cdot x_i^* = p^* \cdot y_j^* \\
    \text{Price correspondence:} & \quad p^* \cdot k_{clc}(x_i^*) = p^* \cdot k_{clc}(y_j^*) \\
    \text{Lipschitzian:} & \quad ||p^*|| \leq C||x||^{1/\alpha} + C||y||^{1/\beta}, \quad \alpha, \beta > 0.
\end{align*}
\]
for all i, $-p^* \in T_i^0$ and for all j, $p^* \in T_j^0$.

We postpone to the end of this section the discussion on the assumptions of the theorem and we now give some examples of cones $T_i$, $T_j$.

The first example of convex cones $T_i$, $T_j$ is given by Clarke’s tangent cones $T_i = T_{P_i}(x^*_i)$ ($i = 1, \ldots, m$), $T_j = T_{y_j^*}(y_j^*)$ ($j = 1, \ldots, n$), which are subsets of $t_{P_i}(x^*_i)$ and $t_{y_j^*}(y_j^*)$, respectively. Theorem 2.1 is in fact a direct consequence of Theorem 3.1 and the proof of Theorem 2.1 given in section 4, will consist in checking that the epi-Lipschitzian assumption implies the above condition (1).

The second example of cones $T_i$, $T_j$ is given by the cone of interior displacements $T_i = k_{P_i}(x^*_i)$ (resp. $k_{c_1 P_i}(x^*_i)$), $T_j = k_{y_j^*}(y_j^*)$, in a situation where they are convex. This is exactly the framework considered by Guesnerie (1975). So, from Theorem 3.1, one deduces the following result, which for finite dimensional commodity spaces, is exactly Theorem 1 of Guesnerie (1975).

**Corollary 3.1.** Let $((x^*_i), (y^*_j))$ be a weak Pareto optimum (resp. a Pareto optimum and assume $m > 1$) such that, for all $i$, $x^*_i \in \text{cl} P_i^*$ and, for all $i, j$, the sets $k_{P_i}(x^*_i)$ (resp. $k_{c_1 P_i}(x^*_i)$) and $k_{y_j^*}(y_j^*)$ are convex. Then, there exists a non-zero price vector $p^*$ in $L^*$ satisfying

(a') for all $i$, $-p^* \in [k_{P_i}(x^*_i)]^0$ (resp. $[k_{c_1 P_i}(x^*_i)]^0$),

(b') for all $j$, $p^* \in [k_{y_j^*}(y_j^*)]^0$.

**Remark 3.1.** We point out that the following inclusion $\text{int} T_{y_j^*}(y_j^*) \subset k_{y_j^*}(y_j^*)$ may be strict, even when $k_{y_j^*}(y_j^*)$ is non-empty, convex, and $\text{int} T_{y_j^*}(y_j^*)$ is non-empty. Hence, one may have $[k_{y_j^*}(y_j^*)]^0 \neq N_{y_j^*}(y_j^*)$ and the conclusion of Corollary 3.1 may be stronger than the one of Theorem 2.1. To show this assertion, consider the production set $Y_i$ in fig. 2, for which $k_{y_j^*}(0) = \{(x, y) | y < \min \{-x, -(1/3)x\}\}$, and $\text{int} T_{y_j^*}(0) = \text{int} \mathbb{R}^2_+$.

Our third and last example of cones $T_i$, $T_j$ is given by $T_i = t_{P_i}(x^*_i)$, $T_j = t_{y_j^*}(y_j^*)$ in a situation where they are convex. So we can state

**Corollary 3.2.** Let $((x^*_i), (y^*_j))$ be a weak Pareto optimum (resp. a Pareto optimum and assume $m > 1$) such that, for all $i$, $x^*_i \in \text{cl} P_i^*$, for all $i, j$, $t_{P_i}(x^*_i)$ and $t_{y_j^*}(y_j^*)$ are convex and

(I bis) \[\begin{cases} \text{either, for some } i, \emptyset \neq \text{int } t_{P_i}(x^*_i) = k_{P_i}(x^*_i) \text{ (resp. } k_{c_1 P_i}(x^*_i)) \text{,} \\
\text{or, for some } j, \emptyset \neq \text{int } t_{y_j^*}(y_j^*) = k_{y_j^*}(y_j^*) \end{cases}\].
Then, there exists a non-zero price vector $p^*$ in $L^*$ satisfying

(a') for all $i$, $-p^* \in n_{p^*}(x^*_i)$,

(b') for all $j$, $p^* \in n_{L^*}(y^*_j)$.

**Remark 3.2.** We also point out that the cones $k_{Y_j}(y^*_j)$, $k_{P^*}(x^*_i)$ (resp. $t_{Y_j}(y^*_j)$, $t_{P^*}(x^*_i)$) may not be convex and the above analysis is no longer valid. This is the case for the following production set, of particular economic importance, $Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0 \text{ and } y_2 \leq \max \{0, -1 - y_1\}\}$ at $y^* = (-1, 0)$ (where it has an ‘inward’ kink). However, it is easily shown to be epi-Lipschitzian and Theorem 2.1 can be applied; in other words, Pareto optima can be ‘supported’ by prices in Clarke’s normal cone, but not in a smaller cone, as in (b') or (b’).

**Remark 3.3.** The following example shows that it is not possible, in Corollary 3.2, to weaken assumption (I bis) by only assuming that, for some $j$, $\text{int } t_{Y_j}(y^*_j) \neq \emptyset$ [or $k_{Y_j}(y^*_j) \neq \emptyset$] or, for some $i$, $\text{int } t_{P^*}(x^*_i) \neq \emptyset$ [or $k_{P^*}(x^*_i) \neq \emptyset$]. In other words, the analogue of Theorem 2.1, with Clarke’s cones replaced by the cones $t_{P^*}(x^*_i)$ and $t_{Y_j}(y^*_j)$, does not hold, in general, even under the assumptions that all these cones are convex. The same example also shows that it is not possible, in general, to weaken assumption (I) by only assuming that either, for some $i$, $\text{int } T_i \subseteq \text{int } t_{P^*}(x^*_i)$, or, for some $j$, $\text{int } T_j \subseteq \text{int } t_{Y_j}(y^*_j)$.

**Fig. 1**

\[ P_1((1, x^2)) = \{(1, x^2 \mid x^2 \geq x^2) \text{ if } x^2 > 1 \]
We consider the following economy with two goods, \( \omega = (2, \frac{1}{2}) \) as the vector of initial endowments, one firm and one consumer, respectively, with a production set \( Y \) and complete, transitive preferences, as represented on fig. 1.

Let \( x_1^* = (1, 1), \ y_1^* = (-1, \frac{1}{2}) \). Then \((x_1^*, y_1^*)\) is a Pareto optimum but one easily sees that \( k_Y(y_1^*) = \mathbb{R}^2 \setminus \{(0, y) | y \geq 0 \} \), hence \( t_Y(y_1^*) = \mathbb{R}^2 \) and \( n_Y(y_1^*) = \{0\} \). Hence, the conclusion of Corollary 3.2 cannot hold. At this point, it is worth stressing that, if \( L \) is finite dimensional, and if \( Y \) is closed, then at every element \( y \) in \( \partial Y \), Clarke's normal cone is not reduced to \{0\}, i.e., \( N_Y(y) \neq \{0\} \) [Clarke (1983, Corollary 2 of 2.5.5)].

Remark 3.4 The result of Theorem 3.1 is no longer true, in general, when the tangent cone \( t_C(x) \) is replaced by Bouligand's tangent cone, \( T_C(x) \), which we recall, is defined for a subset \( C \) of \( L \) and \( x \in \text{cl} \, C \) as follows:

\[
T_C(x) = \{ v \in L | \forall N \in \mathcal{N}, \forall \varepsilon > 0, \exists \tau \in (0, \varepsilon): [x + \tau (v + N)] \cap C \neq \emptyset \}.
\]

Fig. 2
We consider the following economy with two goods, \( \omega = (1, 1) \) as the vector of initial endowments, two firms and one consumer. The production sets of the firms, \( Y_1 \) and \( Y_2 \) are represented in fig. 2. The consumption sets of the consumer is \( \mathbb{R}_2^+ \) and his preferences are represented by the utility function \( u(x^1, x^2) = \min \{x^1, x^2\} \). Let \( x^*_1 = (1, 1) \), \( y^*_1 = 0 \) and \( y^*_2 = 0 \). Then \( (x^*_1, y^*_1, y^*_2) \) is a Pareto optimum but \([\mathbb{T}_{Y_1}(y^*_1)]^0 \cap [\mathbb{T}_{Y_2}(y^*_2)]^0 = \{0\}\).

4. Proofs

We begin with the proof of Theorem 3.1 from which we shall then deduce the other results. We prepare its proof by a lemma.

Lemma 4.1. Let \( C_i, i = 1, \ldots, k \) be arbitrary subsets of \( L \), and let \( x^*_i \) be elements in \( \text{cl} C_i \) (\( i = 1, \ldots, k \)) such that \( \sum_{i=1}^k x^*_i \notin \text{int} C_1 + \sum_{i=2}^k C_i \). Then

\[
0 \notin k_{C_i}(x^*_i) + \sum_{i=2}^k t_{C_i}(x^*_i).
\]

Proof. By contraposition. Let us suppose that \( 0 = v_1 + v_2 + \cdots + v_k \), for some \( v_i \) in \( k_{C_i}(x^*_i) \), and some \( v_i \) in \( t_{C_i}(x^*_i), i = 2, \ldots, k \). Since \( v_1 \) belongs to \( k_{C_1}(x^*_1) \), there exists \( \epsilon_1 > 0 \) and \( N_1 \in \mathcal{N} \) such that

\[
x^*_1 + t(v_1 + N_1) \subset \text{int} C_1 \quad \text{for all } t \in (0, \epsilon_1).
\]

Since the set \( \{(x_i) \in L^{k-1} | \sum_{i=2}^k x_i \in -N_1 \} \) is open and contains 0, there exists \( N_2 \in \mathcal{N} \) such that \( N_2 + \cdots + N_2 \subset -N_1 \). Since, for \( i > 1 \), \( v_i \) belongs to \( t_{C_i}(x^*_i) \), there exists \( \epsilon_i > 0 \) such that

\[
[x^*_i + t(v_i + N_2)] \cap C_i \neq \emptyset \quad \text{for all } t \in (0, \epsilon_i), \quad i = 2, \ldots, k.
\]

Let \( t_0 > 0 \), \( t_0 < \min \{\epsilon_i|i = 1, \ldots, k\} \); from above, we can choose, for \( i > 1 \), \( x_i \) in the set \( [x^*_i + t_0(v_i + N_2)] \cap C_i \) and we let \( x_1 = x^*_1 + \sum_{i=2}^k (x^*_i - x_i) \).

We claim that \( x_1 \) belongs to \( \text{int} C_1 \). Indeed, from above,

\[
x_1 = x^*_1 + t_0 \sum_{i=2}^k (x^*_i - x_i)/t_0 \in x^*_1 - t_0 \sum_{i=2}^k (v_i + N_2)
\]

\subset x^*_1 + t_0(v_1) + N_1 \subset \text{int} C_1.
Hence, we have proved that \( \sum_{i=1}^{k-1} x_i^* = \sum_{i=1}^{k-1} x_i \), \( x_1 \in \text{int } C_1 \), \( x_i \in C_i \) \((i=2, \ldots, k)\), a contradiction with the assumption of the lemma. This ends the proof of the lemma.

**Proof of Theorem 3.1.** Let \( ((x_i^*), (y_j^*)) \) be a weak Pareto optimum (resp. a Pareto optimum and assume \( m > 1 \)) such that, for all \( i \), \( x_i^* \in \text{cl } P_i^* \). We first assume that, for some \( i \in \{1, \ldots, m\} \), say \( i_0 = 1 \),

\[
\emptyset \neq \text{int } T_1 \subset k_{P_1^*}(x_i^*) \quad [\text{resp. } k_{\text{cl } P_i^*}(x_i^*)].
\]

From the definitions of Pareto optimality one deduces that

\[
\omega = \sum_{i=1}^{m} x_i^* - \sum_{j=1}^{n} y_j^* \notin \text{int } P_i^* + \sum_{i=2}^{m} P_i^* - \sum_{j=1}^{n} Y_j,
\]

[resp. \( \omega = \sum_{i=1}^{m} x_i^* - \sum_{j=1}^{n} y_j^* \notin \text{int cl } P_i^* + \sum_{i=2}^{m} P_i^* - \sum_{j=1}^{n} Y_j \)].

But, for all \( i \), \( x_i^* \in \text{cl } P_i^* \) and, for all \( j \), \( y_j^* \in Y_j \), hence, by Lemma 4.1, noticing that, for \( C \subset L \) and \( x \in \text{cl } C \), one has \( t_C(-x) = -t_C(x) \),

\[
0 \notin k_{P_1^*}(x_i^*) + \sum_{i=2}^{m} t_{P_i^*}(x_i^*) - \sum_{j=1}^{n} t_{Y_j}(y_j^*)
\]

[resp. \( 0 \notin k_{\text{cl } P_i^*}(x_i^*) + \sum_{i=2}^{m} t_{P_i^*}(x_i^*) - \sum_{j=1}^{n} t_{Y_j}(y_j^*) \)].

Recalling that, for all \( i \), \( t_{P_i^*}(x_i^*) \) contains \( T_i \) and 0, and, for all \( j \), \( t_{Y_j}(y_j^*) \) contains \( T_j \) and 0, one deduces that

\[
0 \notin T \overset{\text{def}}{=} \text{int } T_1 + \sum_{i=2}^{m} T_i \cup \{0\} - \sum_{j=1}^{n} T_j \cup \{0\}.
\]

But \( T \) is a non-empty open convex subset of \( L \), since \( \text{int } T_1 \) is non-empty (and open), and \( T \) is the sum of non-empty convex subsets of \( L \). Noticing that, if \( T \) is a convex cone (possibly empty), then \( T \cup \{0\} \) is a non-empty convex cone. Consequently, by Hahn–Banach's theorem [Bourbaki (1966, II.5.2., Prop. 1)], there exists a non-zero price vector \( p^* \) in \( L^* \) satisfying
0 \leq p^* \cdot \left( \sum_{i=1}^{m} t_i - \sum_{j=1}^{n} t_j \right) \quad \text{for all } t_i \in \text{int } T_i, \quad t_i \in T_i \cup \{0\} \quad (i = 2, \ldots, m),
\quad t_i \in T_j \cup \{0\} \quad (j = 1, \ldots, n).

Clearly, the above inequality also holds for all \( t_i \in T_i \cup \{0\} \), since \( p \) is continuous and \( T_i \cup \{0\} \subset \text{cl } T_i = \text{cl } \text{int } T_i \) [Bourbaki (1966, II.2.b, Corollary 1)], recalling that \( T_i \) is convex with a non-empty interior. Taking in the above inequality, all the \( t_i = 0 \), \( t_j = 0 \), but one, we deduce that, for all \( i, j \),

\[ 0 \geq -p^* \cdot t_i \quad \text{for all } t_i \in T_i, \quad \text{and } \quad 0 \geq p^* \cdot t_j \quad \text{for all } t_j \in T_j, \]

or, equivalently, \(-p^* \in T_i^0 \) (\( i = 1, \ldots, m \)) and \( p^* \in T_j^0 \) (\( j = 1, \ldots, n \)). We leave to the reader to adapt the above proof to the case where the assumptions of the theorem are satisfied for some firm \( j \). This ends the proof of Theorem 3.1.

At this stage it is worth pointing out the link between our proof and other ones. The proof of Theorem 3.1 is similar to the ones of Guesnerie and Quinzii, whereas Khan and Vohra use a product argument. We point out, however, that Guesnerie (Quinzii) considers only the cone of Dubovickii and Miljutin (resp. the interior of the cone of Clarke) to \( P^*_i \) at \( x_i^* \) (\( i = 1, \ldots, m \)) and to \( Y_j \) at \( y_j^* \) (\( j = 1, \ldots, n \)) since all these cones are assumed to be non-empty. Here, the use of the closed cones \( T_{P_i}(x_i^*) \) and \( T_{Y_j}(y_j^*) \) allows us to make the interiority assumption only for one consumer, or one firm.

Proof of Theorem 2.1. It is deduced from Theorem 3.1 by taking \( T_i = T_{P_i}(x_i^*) \), \( T_j = T_{Y_j}(y_j^*) \) (Clarke's tangent cones) and by checking that these cones satisfy the assumptions of Theorem 3.1. Indeed, for all \( i, j \), \( T_{P_i}(x_i^*) \) and \( T_{Y_j}(y_j^*) \) are convex cones (Proposition 2.1) and \( T_{P_i}(x_i^*) \subset T_{P_i}(x_i^*) \), \( T_{Y_j}(y_j^*) \subset T_{Y_j}(y_j^*) \) (Proposition 3.1). Furthermore, if, for some \( i \), \( P_i^* \) [resp. \( \text{cl } P_i^* \)] is epi-Lipschitzian at \( x_i^* \), i.e., if \( H_{P_i^*}(x_i^*) \) [resp. \( H_{\text{cl } P_i^*}(x_i^*) \)] is non-empty, then, by Proposition 2.1(c), \( \emptyset \neq H_{P_i^*}(x_i^*) = \text{int } T_{P_i}(x_i^*) = \text{int } T_i \) [resp. \( \emptyset \neq H_{\text{cl } P_i^*}(x_i^*) = \text{int } T_{\text{cl } P_i^*}(x_i^*) = \text{int } T_i \)] and, by Proposition 3.1, \( \text{int } T_i = H_{P_i^*}(x_i^*) = k_{P_i^*}(x_i^*) \) [resp. \( \text{int } T_i = H_{\text{cl } P_i^*}(x_i^*) = k_{\text{cl } P_i^*}(x_i^*) \)]. Similarly, if, for some \( j \), \( Y_j \) is epi-Lipschitzian at \( y_j^* \), then \( \emptyset = \text{int } T_j \subset k_{Y_j^*}(y_j^*) \). This ends the proof of Theorem 2.1.

Proof of Proposition 3.1, Part (d). Let \( C \) be a convex subset of \( L \) and \( x \) be an element in \( \text{cl } C \). To prove that the inclusions in (c) are equalities, since
$T_C(x)$ is a closed subset of $L$, it suffices to show that the following inclusion holds:

$$\{\lambda(y-x) | \lambda > 0, y \in C \} \subset T_C(x).$$

Indeed, let $v = \lambda(y-x)$, for some $\lambda > 0$ and some $y \in C$. We recall that $v$ belongs to $T_C(x)$ if for all $N \in \mathcal{N}$ there exist $\varepsilon > 0$ and $N' \in \mathcal{N}$ such that

$$[x'+t(v+N)] \cap C \neq \emptyset \quad \text{for all } x' \in (x+N') \cap C, \quad \text{all } t \in (0, \varepsilon),$$

and we check that the above statement holds for $\varepsilon = 1/\lambda$ and $N' = -\varepsilon N$. Indeed, for $x' \in (x+N') \cap C$ and $t \in (0, \varepsilon)$, since $C$ is convex and $y$ belongs to $C$,

$$x' + t(v - \varepsilon^{-1}(y-x)) = x' + t\varepsilon^{-1}(y-x') \in C$$

$$\quad \in x' + t(v - \varepsilon^{-1}N') \subset x' + t(v + N).$$

Hence $v \in T_C(x)$, which ends the proof of the first part of (d).

We now prove the second part of (d). First, one easily checks from the definitions that

$$H_C(x) = k_C(x) \subset \{\lambda(y-x) | \lambda > 0, y \in \text{int } C\}$$

and the proof of (d) will be completed by showing that

$$\{\lambda(y-x) | \lambda > 0, y \in \text{int } C\} \subset H_C(x).$$

Indeed, let $v = \lambda(y-x)$ for some $\lambda > 0$ and some $y \in \text{int } C$. Then, there exists a neighborhood $N \in \mathcal{N}$ such that $y + N \subset C$. We now choose $N' \in \mathcal{N}$ such that $N' + N' \subset N$. Since $H_C(x)$ is a cone, to prove that $v = \lambda(y-x) \in H_C(x)$, it suffices to show that $y-x \in H_C(x)$ and, from the definition of $H_C(x)$, that

$$x' + t\varepsilon' \in C \quad \text{for all } t \in (0, 1), \quad x' \in C \cap (x+N'), \quad \varepsilon' \in (y-x)+N'.$$

Indeed, let $t$, $x'$ and $\varepsilon'$ as above, then

$$x' + t\varepsilon' = t(x' + \varepsilon') + (1-t)x'$$

$$\in t(x + N' + y - x + N') + (1-t)x'$$

$$\in t(y+N) + (1-t)x' \in C,$$
since \( y + N \subseteq C \), \( t \in (0, 1) \) and \( C \) is convex. This ends the proof of the proposition.

**Remark 4.1.** It is worth noticing that if \( C \) is a convex subset of \( L \) and, either \( C \) has a non-empty interior, or \( L \) is finite dimensional, one further has the equalities

\[
H_C(x) = k_C(x) = \{ \lambda (y - x) | \lambda > 0, y \in \text{int} \, C \} = \text{int} \, T_C(x).
\]

The above equalities are consequences of Proposition 3.1(d) and the fact that \( H_C(x) = \text{int} \, T_C(x) \), either when \( E \) is finite dimensional [Rockafellar (1979)] or when \( H_C(x) \) is non-empty [Rockafeller (1980)] and, by Proposition 3.1(d), \( H_C(x) \) is non-empty if and only if \( C \) has a non-empty interior.

It is also worth pointing out that (even when \( C \) is convex) the inclusion \( H_C(x) \subseteq \text{int} \, T_C(x) \) may be strict [see Counterexample 1 of Rockafellar (1979)].

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