Common Knowledge, Coordination and Rational Limits to Arbitrage

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Summary: We propose a rational framework which generates asset prices that appear irrational. This is accomplished by studying rational expectations equilibria in the presence of two realistic market frictions: immediacy risk and asset-specific orders. We study some properties of such equilibria, in particular the prevalence of arbitrage and of informational inefficiencies. Trading within an institution becomes decentralized into trading desks because the market frictions imply that higher order uncertainty remains unresolved. The resulting imperfect coordination and lack of common knowledge between traders prevents them from fully exploiting arbitrage opportunities and gains from trade.

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1 Introduction

Common knowledge (e.g. Aumann (1976)) and rational expectations (as ex- p osed by Muth (1961) and subsequently refined by Radner (1967), Lucas (1972), Grossman (1977) and Radner (1979)) have become the standard pil lars to modelling trade and asset pricing. Recall that an event $E$ is common knowledge between a group if all know $E$, all know that all know it, all know that all know that all know it, and so on ad infinitum. It has, however, be come clear that the frictionless world with common knowledge and rational expectations puts some strong restrictions on the characteristics of equilibria, and it has proved hard to reconcile some stylized features of markets with the frictionless version of this standard model. Among them, we would like to mention two.

First, the standard assumptions have led to the remarkable no-trade theorems pioneered by Milgrom and Stokey (1982), which are in stark contrast to the large trading volumes observed in the markets which cannot plausibly be linked to the exploitation of gains from trade. Dow, Madrigal, and da Costa Werlang (1990) further strengthen this result. This failure to explain the occurrence of trade has led researchers to relax some of the assumptions in Milgrom and Stokey. Several models rely on the axiomatic approach to common knowledge to introduce “bounded rationality” which breaks the no-trade theorem, among them Geanakoplos (1988) and Samet (1990). Neeman (1996) relaxes the common knowledge assumption of rationality, but keeps rationality of traders a well-defined event, and shows that trade occurs. More recently, behavioural finance imposes biases directly upon the agents’ behaviour in order to generate repeated trade (e.g. Scheinkman and Xiong (2003) impose over-confidence).

Second, the standard frictionless model (with price-takers, but with or without complete markets) leaves no room to mispricings or arbitrage opportunities at an equilibrium, for the agents’ strategies at an arbitrage opportunity are incompatible with market clearing. Real world markets do, however, allow occasional mispricings, arbitrage opportunities and informational inefficiencies. Similarly, the standard model assumes away the fact that in reality, trading and portfolio decisions are taken in a distributed environment by a variety of different agents in real time, which creates well-known coordination problems. In fact, it is commonly argued that distributed knowledge (intuitively, knowledge is distributed when a set of agents knows something payoff relevant to all that no single agent knows) is an apt description of the
knowledge conditions in which modern firms are increasingly finding themselves in by force of the rapid rate of innovation and the trends pointing towards a more knowledge-based economy. For instance, in the strategy field, Tsoukas (1996) conceptualizes the firm as a distributed knowledge system, and Granstrand, Patel, and Pavitt (1997) document the increasing extent to which the knowledge bases controlled by major technology-intensive corporations are distributed. Lessard and Zaheer (1996) discuss the implications of distributed knowledge for decision-making and provide an application to currency management. Foss and Foss (2002) relate distributed knowledge to authority.

This paper addresses both issues by imposing realistic trading restrictions upon agents’ actions within a non-strategic price-taking setup. Agents are assumed to be fully rational, we do not need either irrationality, bounded rationality or behavioural biases to derive our results. Agents within each financial institution optimally coordinate given the reality of their environment, and submit their demand functions, whereupon an electronic auctioneer computes the market-clearing prices, if they exist. By rational expectations, this price coincides with the price the agents expected to see in that state. The resulting excess demand quantities determine the equilibrium allocation. The objects of choice in our model can be either consumption goods or assets, but we shall mostly focus on assets. Indeed, it is an empirical market microstructure fact that one cannot condition one’s demand for a given asset upon the prices of other assets. In financial markets in particular, the limit orders for the stock of IBM cannot be made contingent upon the price of any other stock or random variable. We refer to this restriction as “asset-specificity.” ¹ This is in contrast to the nearly exclusive reliance in economics upon Marshallian demand schedules which effectively allow the demand for any good or asset to depend upon the whole price vector. In a sense, real financial markets are more decentralized, with multiple trading posts, than standard theory assumes, and it is this feature that gives rise to the coordination problem among traders. As stated in the pioneering article by Hayek (Hayek 1945), p.524),

If we …agree that the economic problem of society is mainly one

¹More generally, one could assume that the set of all commodities, goods or assets, is partitioned, and that the elements of a partition \( k \) may depend only upon the price vector of the subset of elements in partition \( k \). But for financial realism, we assume that the partition corresponds to a collection of singletons.
of rapid adaptation to changes in the particular circumstances of time and place, ... decisions must be left to the people who are familiar with these circumstances, who know directly of the relevant changes and of the resources immediately available to meet them. We cannot expect that this problem will be solved by first communicating all this knowledge to a central board which, after integrating all knowledge, issues its orders. We must solve it by some form of decentralization.

In the formal model, we assume that the mapping from states to endowments, preferences and asset payoff dependent signals is common knowledge, but that each agent only observes his own preferences and endowments, as well as his own signal about the value of the asset. As in Diamond and Verecchia (1981), information is diffuse and gets aggregated at the equilibrium. Since agents don’t exactly know the realized trading environment, they cannot predict prices with certainty. If the agent could submit Marshallian demand schedules, the uncertainty about the trading environment, or about prices (its payoff-relevant portion), is inconsequential since ex-post the submitted demand function maps the realized price vector into the same allocation as in a world with a known trading environment (this follows from Proposition 1 below). Having observed his preferences, endowments and signal, agent h’s information set can be represented by the information sets generated by that observation. Asset-specificity requires that agent h submits a demand for asset a that is measurable with respect to his private information and to the information generated by the observed price of asset a, and the same applies to all other assets as well. Demand cannot depend on information that would be revealed by observing the price vector of the assets other than a. Whereas the Marshallian model is equivalent to a model where the entire trading environment is common knowledge, the knowledge of each agent in an asset-specific economy becomes endogenously segmented into non-comparable information sets, depending on prices. It follows that we are able to endogenously determine a trading environment within which different traders have different priors.

The aim of much of the literature is to relax the sometimes paradoxical implications of the excessively strong assumption of common knowledge. Here the uncontroversial real-world requirement that demand only depends on the price of the commodity demanded breaks the common knowledge of the payoff-relevant event, not because different agents now necessarily have
different information, but because each investor’s decision problem becomes a team decision problem. Marschak (1955) has called a team “a group of persons each of whom takes decisions about something different but who receive a common reward as the joint result of all those decisions” (i.e. there are no conflicts of interest between members). In our setup, each commodity will effectively be traded by a team-member with an (endogenous) non-comparable information set that differs from the one of any other team-member of the same agent because it has been refined by the information revealed by the price that this trader is the only one within the team to observe. The reason why the problem is isomorphic to a team problem is that when the investor submits his demand function for asset $a$, he programs the following information extraction problem into the schedule: if the realized price is $q_a$, then team member $a$ refines his information by the information revealed by $q_a$ and chooses a quantity that depends on what the team member believes the realizations of the other prices were and therefore what the quantities traded of all other members were.

By the fact that prices cannot be communicated across team members, the configuration of knowledge becomes what is known as distributed knowledge (Halpern and Moses (1990)). When all members of agent $h$ share (“join”) their information, any payoff-relevant event is commonly known by them, but individually their payoff-relevant information is partial only. An event can be mutually known (meaning that each team member knows the event is true), or even known to a higher order (where each team member knows the event is true, and knows that each member knows it to be true etc, but not ad infinitum), but it may not be commonly known. Another way of putting this is to say that we derive different priors within our model from the observed microstructure of financial markets.

Each of the members of an agent’s team will try to implement the trades that maximize the agent’s utility function, but coordination between team members is limited by the information about all prices revealed or transmitted by any one price. It is well-known that the absence of common knowledge forbids perfect coordination (Halpern and Moses (1990), Fagin, Halpern, Moses, and Vardi (1995)). But given that the traders are members of the same team, they do not act strategically. It follows that one can have a Bayesian maximization approach in imperfect communication that gives rise to optimal, but imperfect, coordination, without knowledge being common (also see Morris and Shin (1997)).

In this paper we characterize the optimal rules of action and describe
the resulting rational expectations equilibrium, or REE. We show that assetspecificity may generate trade where the standard model wouldn’t. In Milgrom-Stokey fashion, if the allocation is ex-ante Pareto efficient, the arrival of new information will not induce further trade if it is common knowledge when a trade takes place that it is feasible and individually rational among agents. Such common knowledge among agents cannot be expected if knowledge is distributed among the different team-members of each agent in a non-comparable fashion, therefore enabling trade. We also show that the resulting decentralized decision structure and the resulting non-strategic coordination problem lead to team-member specific pricing. In other words, while in standard trading environments the marginal rates of substitution of any agent correctly prices all assets, here asset a is only priced correctly when using the marginal rates of substitution of the team members (h, a) trading asset a, all h. This pricing dislocation has the potential to generate arbitrage opportunities and other inefficiencies. Arbitrage opportunities can survive at the market-clearing equilibrium because even if the concerned team members know there is an arbitrage, they may be unsure whether all implement the strategy faultlessly. The arbitrage survives even if the team-members have higher-order knowledge about the opportunity: each traders knows that each trader knows that each trader knows, but not ad infinitum. Unless the event is common knowledge, there remains some ambiguity in each team member’s mind as to whether the other team members execute their legs of the arbitrage operation correctly, which, together with risk-aversion, bounds the demands and allows markets to clear.

Related Literature. There is a large organizational theory literature on efficient organizational structuring so as to best solve informational problems, Simon (1947), March and Simon (1958), Mintzberg, Raisinghani, and Théoret (1976), Bolton and Dewatripont (1994), Garicano (2000) and Vayanos (1999). Recent works by Radner (e.g. Radner (1993)) and Van Zandt (e.g. Van Zandt (1999)) focus on organizations as information processors. In our paper, the optimal structure follows directly from the market microstructure, and we do not impose further costs or bounds to information processing that would call for a certain hierarchical structure.

The investors’ optimization problem in this paper does have the same team flavor and bears the same coordination issues among team members that was analyzed by Marschak (1954), Marschak (1955), Radner (1955), Radner (1962) and Marschak and Radner (1971). In particular, Marschak
posed a problem similar to the one studied here. There are two partners in an arbitrage firm. Partner 1 observes the ask price while partner 2 observes the bid price, and they can take one of three actions: commit the firm, do nothing and finally communicate, which comes at a cost. Under some assumptions, a solution to the problem can be found in Kiefer and Orey (1953) (also see Beckmann and Waterman (1953)). Some papers in management science deal less with information per se as with the related problems of strategic decision making in environments where various production centers have idiosyncratic information that, although instantaneously transferrable, is private and requires incentive contracts to induce revelation, e.g. transfer pricing. See for instance Harris, Kriebel, and Raviv (1982) or Jensen and Meckling (1992).

This paper is also related to the (mostly computer science and engineering) literature on distributed systems following the seminal article by Halpern and Moses (1990). A brief survey of the large body of engineering research of the so-called "Cooperative Multiagent Systems," in the past called "Distributed Artificial Intelligence," as well as their implementations, can be found for instance in Lesser (1999). But coordination has recently received a wide treatment particularly in the study of games. Our notion of non-strategic coordination is in contrast with the one in strategic games (see e.g. Rubinstein (1989), Monderer and Samet (1989)). There higher (but finite) knowledge may not induce any coordination that would lead to the optimal equilibrium. Relatedly, the coordination problem has been revisited by Carlsson and Van Damme (1993) and Morris and Shin (1998). Shin (1996) raises the related question as to how vulnerable different trading systems are to higher order uncertainty about the fundamentals.

The REE aspects that occur here have been analyzed, in different forms, among others, by Green (1977), Grossman (1977), Radner (1979) and Allen (1981). The aggregation of distributed information by prices in this paper is a generalization of Diamond and Vercocchia (1981). The only papers on arbitrage and information transmission we are aware of are Fremault (1993) and Cornet and De Boisdeffre (2002).

A growing body of literature studies how arbitrage opportunities can survive. In a rational environment, Dow and Gorton (1994) introduce trading frictions in order to limit the trading capabilities of arbitrageurs. Zigrand (1999) relies upon restricted participation together with strategic arbitrageurs. Strategic arbitrageurs choose not to exploit the opportunities fully since this would lead to a narrowing of their margins. De Long, Shleifer,
Summers, and Waldmann (1990) and Shleifer and Vishny (1997), and subsequently others, rely on bounded rationality, noise traders and short horizons. Behavioural finance is also to a certain extent aiming to provide a rationale for limits to arbitrage, since otherwise many of the asset pricing patterns introduced by behavioural biases might be arbitraged away by rational arbitrageurs. Refer for instance to the survey by Barberis and Thaler (2003).

Structure of the Paper. The structure of the economy is presented in Section 2. The two innovative market microstructure assumptions used in this paper are described in Section 3. Section 5 introduces the investor’s optimization problem, and Section 4 deals with the characterization of the rational expectations equilibrium. It also contains most main results. Section 7 concludes. All proofs are relegated to the appendix.

2 The Economy

As mentioned in the introduction, individuals face two layers of risk in this economy, first about the fundamentals of the trading environment, the state of information, and then about the precise realization of endowments and asset payoffs, the state of nature. At time zero they face uncertainty about the fundamentals, i.e. the endowments and preferences of market participants, which translates into uncertainty about prices. The asset demand functions they submit are not required to be either market or limit orders, but they are required to only depend on their own price, i.e. they need to be asset-specific. For instance, the demand schedule for asset $a$ is a function of the price of asset $a$, $q_a$, only, as opposed to the whole price vector $q$. At time one this uncertainty is resolved, asset orders get executed and the households consume. They are, by the very nature of immediacy risk, unable to re-trade at the equilibrium price. The state of nature is realized at time two and final consumption occurs in the realized state. It might be convenient to visualize investor $h$’s utility maximization program in the following fashion. Given that asset specific demand functions have to be handed over to the auctioneer before the state of information is realized, every investor’s demand function for some asset $a$ can be thought of as being represented by a team member $(h,a)$ having the same preferences as the investor, and with the restriction that the $A$ members cannot communicate, so that $(h,a)$’s private informa-
tion (over and above the private information arising from h’s endowments and preferences) corresponds to the observation of the price of asset a, qa.

2.1 General Setup of the Economy

There are H investors in this economy, H assumed to be a finite set. Investors consume the unique consumption commodity, which also serves as the numéraire, each date \{0, 1\} and at date 1 in each state s ∈ S. Each investor h is endowed with \( w^h := \{w^h_0, w^h_1, \ldots, w^h_S\} \in \mathbb{R}^{S+1} \) units of numéraire. At time 0, investors trade in \( A \leq S < \infty \) assets. Assets pay off at date one. The payoff matrix \( R \) is of dim \( S \times A \). Row \( s \) of \( R \) is denoted by \( d_s = (d_{1,s}, \ldots, d_{A,s}) \in \mathbb{R}^A \). \( d_{a,s} \) represents the numéraire payoff of asset \( a \) in the state of nature \( s \). Also, assume there is a portfolio \( y \in \mathbb{R}^A \) such that \( R y \gg 0 \). This assumption assures that not all prices are no-arbitrage prices, see Geanakoplos and Polemarchakis (1986). For instance, it rules out \( R = \begin{bmatrix} 1 & \end{bmatrix} \) and \( R = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \), i.e. economies where no asset and no combination of assets can be used as a “safe asset” to carry purchasing power over to the next period.

The consumption set is \( X^h := \mathbb{R}^{S+1} \). For easy comparison to standard models, utility functions are assumed to be of the Von-Neumann-Morgenstern type \( U^h := u^h_0 + \sum_s p_s u^h_s \). \( p \) represents the investors’ common prior. For \( s = 0, 1, \ldots, S \), \( u^h_s \) is assumed to be smooth, strictly differentiably concave, and is differentiable monotonic, by which we mean that \( u^h_k(x) \geq k > 0 \) for all \( x > 0 \), \( s = 0, \ldots, S \).

So far the setup is entirely standard. The randomness of asset prices will be a major ingredient in our study. In period zero, investors learn about their environment, in particular about their own preferences and endowments, and they may receive an asset payoff relevant signal as well. They do not learn directly about any other investor’s preferences, endowments or signals. In this we follow Diamond and Verrecchia (1981), except that we allows investors to update their information about their trading environment since their own characteristics may partially reveal information as to other investors’ characteristics. Learning matters in our setup for the additional reason that prices are not known before the trade and that the demands submitted to the auctions for the various assets have to be asset-specific. Investors consume at time zero at subsequently they consume their portfolio proceeds together with their endowments at time one in one of the possible states of nature.
$(\mathcal{E}, \mathcal{F}, \mu)$, with $\mathcal{E} \subset \mathbb{R}^E$ a compact Borel set serving as the support of $\mu$, denotes the complete probability space of the parameters describing the states of information $\epsilon$, such as the endowments and preferences of all investors across states and time, as well as signals about the likelihoods of the various states of nature in $S$ chosen by nature. The state of nature represents the realisation of endowments and the state-dependent parameters affecting preferences.

For simplicity, assume that $\epsilon := \{\epsilon^h\}_{h \in H}$, with $\epsilon^h := \{w^h, \pi^h, \xi^h\}$. The vector $\pi^h \in \mathbb{R}^P$ is a finite dimensional vector of parameters affecting the utility function, and $\xi^h \in \mathbb{R}^S$ is an asset-payoff relevant signal that allows investors to update their prior probabilities of the various states of nature $p$. It follows that $E = H[(S + 1) + P + S]$. It is important to realize that no results in this paper are qualitatively affected if either endowments or preferences or signals were common knowledge, as long as at least two dimensions of uncertainty remain unresolved for each investor when markets open. We shall return to this observation. When $\epsilon$ is realized, each investor $h$ is informed of the realization$^2$ of $\epsilon^h$ but ignores the realization of $\{\epsilon^k\}_{k \neq h}$. $\mathcal{F}^h$ is the information generated by $\epsilon^h$ and describes investor $h$'s information after having learned his own parameters. We shall formally define $\mathcal{F}^h$ below.

The major innovation of this paper is to require that prices are determined in an auction. Investors therefore need to submit their demand functions prior to observing the market clearing prices. But just as in actual financial markets, the demand function for some asset $a$ cannot depend on information revealed by prices other than by the price of asset $a$, $q_a$. In other words, the demand schedule for asset $a$ must be measurable w.r.t. the information $\mathcal{F}^{h,a}$ available after having observed $\epsilon^h$ and $q_a$. We can interpret $\mathcal{F}^{h,a}$ as the information available to investor $h$'s trader specialized in asset $a$, $(h, a)$. We also refer to $(h, a)$ as a member, trader or agent of team $h$. Investor $h$’s demand function for asset $a$ is denoted by $\theta^h_a = f^h_a(q_a, \epsilon^h)[\phi]$, where $\phi$ is the investor’s forecast function:

Definition 1 A price function or a forecast function (Radner (1979)) \(\phi\) is any $\mathcal{F}/\mathcal{B}^A$-measurable mapping from the set of states of information $\mathcal{E}$

$^2$This is similar to the setup in Green (1977). In Allen (1981) and Allen (1985a), after the realization of the state of information, investors are assumed to know their endowments and prices but they find out about their preferences from prices. In her model, that’s the only incentive to learn from prices. Here investors are assumed to know their endowments and preferences, and use that knowledge to predict prices.
to the space of asset price vectors, a subset of \( \mathbb{R}^4 \).

By a REE we then mean:

**Definition 2** A Rational Expectations Equilibrium is a price function \( \phi \) and asset demands \( \{ f^a_h \}_{a \in A, h \in H} \) solving the investors’ problems, such that markets clear and forecasts are confirmed:

\[
\sum_{h \in H} f^a_h(\phi(\epsilon), \epsilon^h)[\phi] = 0 \quad (\text{almost all } \epsilon \in \mathcal{E}), (\forall a \in A)
\]

Commodities markets then clear by Walras’ Law.

It is also useful to introduce some standard terms in order to characterize knowledge and information across team members. While with finite state spaces the definition of “knowledge” is unambiguous, this is not so with infinite state spaces. Various authors have ported Aumann’s (Aumann (1976)) concept of common knowledge to general state spaces using different, though related, definitions of knowledge. For our purposes, any one of the definitions below can be chosen.\(^3\)

First, information and knowledge can be thought of in terms of information partitions, as in Aumann (1999). Let \( I^{h,a} \) be the information function of member \((h, a)\), i.e. \( I^{h,a}(\epsilon) \) is the set of all states that \((h, a)\) cannot distinguish from \( \epsilon \). It can be shown that the information functions generate a partition of \( \Omega \), denoted by \( \mathcal{I}^{h,a} \). Relatedly, the knowledge operator captures when an agent knows a certain event. Formally, \( K^{h,a} : \mathcal{F} \to \mathcal{F} \), and for any event \( E \in \mathcal{F} \),

\[
K^{h,a}E := \bigcup \{ \delta \in \mathcal{I}^{h,a} : \delta \subset E \}
\]

The universal field of \( \mathcal{I}^{h,a} \), denoted by \( \mathcal{P}^{h,a} \), is the family of all unions of events in \( \mathcal{I}^{h,a} \). Note that \( K^{h,a}E \) is in the universal field of \( \mathcal{I}^{h,a} \), but it

\(^3\)Nielsen (1984) (also see Allen (1983)) introduces yet another definition of knowledge and common knowledge, namely directly in terms of Boolean sigma-algebras. Since the event that \((h, a)\) knows some event \( E \in \mathcal{F} \) is not well-defined in terms of sigma-algebras, Nielsen focuses on generalized events instead. A generalized event is the equivalence class of the event \( E \), i.e. the set of events \( A \) s.t. \( (A \setminus E) \cup (E \setminus A) \) is null. Assuming that \( \mathcal{I}^{h,a} \) is the complete \( \sigma(\epsilon^h, \phi_a) \), denote by \( \Sigma(\mathcal{I}^{h,a}) \) the Boolean sigma-algebra of generalized events. We can then say that the generalized event that \((h, a)\) knows the generalized event \( \tilde{E} \) is \( K^{h,a}\tilde{E} := \bigcup \{ \delta \in \Sigma(\mathcal{I}^{h,a}) : \delta \subset \tilde{E} \} \). The advantage of using this Boolean sigma-algebra of generalized events rather than the universal field is that there is a well-defined probability function \( \tilde{\mu} \) on a Boolean sigma-algebra, but not on a universal field. We shall not exploit this approach in the sequel.
may not be itself an event, i.e. belong to $\mathcal{F}$. This is a sense in which a decision maker can decide whether a set is true, without necessarily being able to make a coherent probabilistic judgment about the set. Refer to Dubra and Echenique (2001) for a concrete example. In that sense we say that knowledge, or information, is $\mathcal{F}^{h,a} = \mathcal{P}^{h,a}$.

Second, Brandenburger and Dekel (1987) provide a “Bayesian” definition of knowledge in terms of posterior distributions. For each agent $(h,a)$, fix a version of the regular conditional distribution function $Q^{h,a} : \mathcal{F} \times \mathcal{E} \to [0,1]$. The event that agent $(h,a)$, with sigma-field $\mathcal{F}^{h,a} = \sigma(e^h, \phi_a)$, knows $E$ at $\epsilon$ is:

$$K^{h,a}E := \{ \epsilon \in \mathcal{E} : Q^{h,a}(\epsilon) = 1 \}$$

The authors also study some of the relationships between the various definitions of common knowledge and provide technical conditions that guarantee their equivalence.

Define, for $A' \subseteq A$, the operators $K^h_{1,A'} E := \cap_{a \in A'} K^h_a E$, and $K^h_{m+1} := K^h_{1,A'} K^h_{m,A'} E$. Common knowledge is then defined as follows. An event $E$ is common knowledge between $\{(h,a)\}_{a \in A'}$ in the state $\epsilon \in \mathcal{E}$ if $\epsilon \in K^h_{\infty,A'} E := K^h_{1,A'} E \cap K^h_{2,A'} E \cap \ldots$. In words, $E$ is commonly known among the group $A'$ of traders of investor $h$ iff all traders know that all traders know that all traders know that all traders know ad infinitum.

For some of the purposes in this paper, any one of the definitions of knowledge $K^{h,a}$ given above can be used. But since arbitrage is defined as a sure gain, it may be more natural to talk about arbitrage in terms of partitions. We therefore define the set of admissible equilibrium pricing functions via partitional knowledge. On the other hand, when investors devise optimal portfolios in the absence of exploitable arbitrage opportunities, they are Bayesian maximizers, and the definition of Brandenburger and Dekel (1987) may be more appropriate. But this makes no substantial differences in this paper, and we occasionally provide conditions for both knowledge operators. Having reviewed the main ingredients, in the next two sections we elaborate upon the two assumptions that we consider innovative and that play a crucial role in the results derived.
3 The Roles of Immediacy Risk and of Asset-Specific Orders.

In many auction-based or order-driven financial markets (as opposed to dealer markets where dealers sometimes offer firm quotes for a specified depth), the price paid or received is not known at the time the order is submitted, and the seminal models Kyle (1985) and Kyle (1989) have captured this feature when traders are strategic. The resulting allocation problem also bears resemblance with Lucas (1990). In auction-based markets investors never explicitly observe prices first and then subsequently update their beliefs and demands. The information refinement needs to be directly integrated into the submitted schedules. This introduces price uncertainty at the time the investors submit their orders. Investors with rational expectations know the price distribution, but they cannot be exactly sure at which prices their orders will be executed. This we call *immediacy risk*. The traditional frictionless economy where agents have rational expectations and observe prices before choosing their optimal actions is called a *Radner economy*.

Next, let us depart from the Radner model in a different direction by adding the constraint upon investors that their demand function for asset \(a\) may only be explicitly conditioned on its *own* price, rather than on the entire price vector. We call this restriction *asset-specificity*. Incorporating any one (and only one) of those realistic market microstructure features into a standard Radner economy will not alter the REE:

**Proposition 1** If, in a Radner economy, we relax

1. either the assumption that agents can submit their excess demand after having observed prices,

2. or the assumption that the demand for an asset can be conditioned on prices of other assets

then the resulting REE still coincides with the REE in the original Radner economy, \(\phi = \phi^R\).

When both assumptions are relaxed simultaneously, REE typically differ from the Radner equilibrium, at least if \(\mathcal{E}\) is uncountable. The reason for \(\mathcal{E}\) uncountable is in fact very simple. If the state space is finite or countable,
we can show that the team member \((h, a)\) can typically induce the payoff relevant joint signal from only observing the price of the single asset \(a\) (and the same holds for all assets \(a \in A\)), i.e. typically any single price \(q_a\) is fully revealing:

**Lemma 1 (Discrete States)** When the state of information is a discrete random variable, relaxing the combined assumptions of asset-specific orders and of immediacy risk generically (in \(\epsilon\)) does not change the Radner REE, \(\phi = \phi^R\).

Loosely speaking, however, investors have to be a bit ‘more precise’ because they have to be able to induce the joint signal from a single price, rather than from a price vector.\(^4\) Typically, this would no longer be true when one asset price does not reveal all payoff relevant information. In fact, Proposition 5 below shows that, for a dense and full measure set of economies, if \(\phi\) is smooth, then observing one price leaves the agent with \(E - 1\) dimensional residual uncertainty.

## 4 Arbitrage and Admissible Pricing Functions

It is well-known that in standard economies the image of \(\phi\) is a subset of \(Q\), the set of no-arbitrage prices defined as follows:

**Definition 3** A price vector \(q\) admits **No Arbitrage (NA)** if there is no trade \(y \in \mathbb{R}^A\) such that

\[
\begin{align*}
\text{either} & \quad y'q \leq 0 \quad \text{and} \quad Ry > 0, \\
\text{or} & \quad y'q < 0 \quad \text{and} \quad Ry \geq 0
\end{align*}
\]

The set of such vectors is denoted by \(Q\).

The admissibility requirement for \(\phi\) in a Radner economy is then

\[\Phi^R := \{\phi^R : \mathcal{E} \to \mathbb{R}^A, \mathcal{F}/\mathcal{B}^A\text{-measurable, such that } \phi^R(\mathcal{E}) \subset Q\}\]

\(^4\) One device that would preclude full revelation even with a discrete state space is to restrict the set of possible prices to a discrete set, i.e. to impose “ticks.” We do not pursue this idea here.
This is overly restrictive in our setup where investors’ schedules need to be asset specific and submitted before prices are known. In such a setting, the demand schedule for asset $a$ needs to be a Baire function $\alpha_a$, i.e. a function from $\mathbb{R} \rightarrow \mathbb{R}$ which is $\mathcal{B}/\mathcal{B}$ measurable. Call the set of $\mathcal{B}/\mathcal{B}$ Baire functions by $\hat{\mathcal{B}}$, and denote for a given profile $\alpha^h$ the sets $M(\alpha^h) := \{ \epsilon \in \mathcal{E} : (R\alpha^h > 0 \text{ and } \phi \cdot \alpha^h \leq 0) \text{ or } (R\alpha^h \geq 0 \text{ and } \phi \cdot \alpha^h < 0) \}$ and $\bar{M}(\alpha^h) := \{ \epsilon \in \mathcal{E} : (R\alpha^h \geq 0 \text{ and } \phi \cdot \alpha^h \leq 0) \}$. Admissibility is then\footnote{Given the assumption that there is a $\bar{y}$ for which $R\bar{y} > 0$, there is no arbitrage according to Definition 3 iff it is true that whenever $Ry > 0$, we have $q_{yR} > 0$. The Bayesian equivalence to $K^h M(\alpha^h) = \emptyset$ is the requirement that $Q^h(\alpha^h : \phi > 0 | R\alpha^h > 0) > 0$ a.s., where we denote, for a given $\alpha^h \in \mathbb{B}^A$, the regular conditional probability of an event $E$ given $\sigma(e^h, R\alpha^h > 0)$ by $Q^h(E | R\alpha^h > 0)$.}

$$
\Phi := \left\{ \phi : \mathcal{E} \rightarrow \mathbb{R}^A, \mathcal{F}/\mathcal{B}^A\text{-measurable, such that} \right. $$
\[ \cup \{ \delta \in \mathcal{I}^h : \delta \subset N \} = \emptyset \quad \text{whenever} \]
\[ N \subset \bar{M}(\alpha^h) \text{ and } N \cap M(\alpha^h) \neq \emptyset , \forall \alpha^h \in \mathbb{B}^A, \forall h \in H \}
$$

This formulation of $\phi$ as a pricing function such that no investor can submit asset-specific schedules that pay off (weakly) in all dates and states of nature may not reveal the thought processes that investors have to go through to establish $\alpha^h$. Instead, let us now focus on the interim period in which prices are realized and agent $(h, a)$ knows both $\epsilon^h$ and $q_a$ (but not $q_b$, $b \neq a$). The reason the Radner admissibility requirement is too strong is the fact that even if $\phi(\epsilon)$ allows for arbitrage, this may not prevent it from being an equilibrium since no investor may realize that there is an arbitrage (for instance as defined in Definition 3, which makes no reference to information sets). For instance, assume that $\phi$ is such that at $\phi_a(\epsilon)$ team member $(h, a)$ knows for sure that there is an arbitrage strategy involving asset $a$. This does not necessarily cause the price system to be nonviable, since the other legs of the operation may not realize there is an arbitrage. Worse, suppose all legs of the arbitrage trade know there is an arbitrage at $\phi(\epsilon)$, but that one or more of them do not know that the others know it. Or it could be that it is in fact common knowledge (CK) that there is an arbitrage, but that traders cannot coordinate on one of the possible opportunities. In short, the team members will not be able to fully take advantage of an opportunity unless the exact opportunity is CK among the relevant subset of team members. Absent CK, traders will only be able to take advantage of what in effect amounts to a
very good risk-arbitrage deal. Due to risk-aversion and to the possibility that the submitted demands do not actually form an arbitrage portfolio, their decision problem is well-defined and a solution exists. The section below contains a number of illustrative examples that show that the set of admissible pricing functions $\phi$ is much larger than the set in standard Radner economies. Even price realizations that are far inside the set of arbitrage prices may be compatible with a REE since the apparent opportunity is not common knowledge between the traders, which means that they do not unambiguously coordinate their trades upon the opportunity.

4.1 Trading Desks, Limits to Arbitrage and Risk-Arbitrage

This phenomenon mirrors the real-world problems that large trading desks with specialized traders face. Since no trader is able to follow all asset prices all the time, arbitrage opportunities are not fully exploited. Traders do coordinate imperfectly and are therefore able to take advantage of some opportunities to a certain degree. These trades are viewed as being good deals, but the team members do not realize the full profits they could have made had it been possible to coordinate perfectly and trade simultaneously.

An arbitrage is defined as a sure gain. Therefore, arbitraging requires perfect coordination (at least in order to take advantage of those nontrivial opportunities that require strategies involving more than one asset). But perfect coordination is impossible exactly because perfect coordination requires common knowledge (as shown by Halpern and Moses (1990), Fagin, Halpern, Moses, and Vardi (1995) and Morris (1995)), which is ruled out by the joint market microstructure assumptions (and the assumption that the state-space be uncountable). The following is adapted from Halpern and Moses (1990):

Proposition 2 (Arbitrage Trade Requires CK of Arbitrage) Any correct protocol for the arbitrage problem has the property that whenever the traders execute an arbitrage, it is CK between them that they are arbitraging.

To see why this is true, take the case of a group of two traders. Assume that both execute an arbitrage trade at a state $\omega = q$. Trader 1 executes the same trade on asset 1 in any state $\omega' \in I^{h,1}(\omega)$. By the correctness of protocols, his action is an arbitrage also at $\omega'$, and the same is true for trader 2 at $\omega'$. So both arbitrage at $\omega' \in I^{h,1}(\omega)$. This is true for any $\omega' \in I^{h,1}(\omega)$,
and for any \( \omega \) at which the protocols execute an arbitrage trade. It follows that both traders know they arbitrage whenever a state occurs at which their protocols have an arbitrage trade programmed up. The set of such states is therefore self-evident (see definition in Footnote 6 below). We can conclude that the existence of an exploitable arbitrage implies that it is CK between the traders that they are arbitraging. Common knowledge is a prerequisite for arbitraging.

Arbitrage opportunities that are common knowledge and allow perfect coordination cannot occur at an equilibrium by market clearing, since such trades are programmed into the submitted excess demand schedules. Admissible price functions therefore cannot allow information refinements that would lead logically competent traders to situations at which an arbitrage is CK. The impossibility of perfect coordination at a given equilibrium price realization, no matter what (finite) depth of knowledge the team members have or what level of common \( p \)-belief obtains (for \( p < 1 \)), allows their price-taking optimization problem to have a solution. In other words, opportunities that ex-post, or to an outside observer, look like pure arbitrages, really may only represent risk-arbitrages to the investors. Therefore the market microstructure assumption of asset-specificity provides a simple and endogenous mechanism to guarantee incomplete arbitraging. This mechanism could also be used in behavioural models to prevent the arbitrageurs from arbitraging the biases away.

To illustrate these ideas within our model, assume that the first asset pays off 1 in the first state only, while the second asset is a riskless bond, \( R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). The absence of arbitrage corresponds to the requirement that \( q_2 > q_1 > 0 \). For simplicity assume that the posterior price distribution of investor \( h \) admits a density whose support is \( \phi(E) \). Intuitively, while the Bayesian investor and his traders may learn about the environment and update his posterior, his signals do not allow him to exclude events from happening. It follows that we can think of knowledge either along the lines of information functions and sigma-algebras or along the lines of sets to which agents attach unitary conditional probability, see Brandenburger and Dekel (1987). As a convention, on the figures below, a representative price realization is represented by a bold point, and the support of the density of prices is given by the boxed area. For simplicity, introduce the induced probability space of asset prices be \( (\Omega, \mathcal{B}^A, \mathbb{P}) \), where \( \Omega = \phi(E) \), \( \mathcal{B}^A \) is the A-dimensional Borel sigma-algebra, \( \mathbb{P}(B) = \mu \circ \phi^{-1}(B) \) for \( B \in \mathcal{B}^A \), and where \( \omega \equiv q \). The set of possible realizations of the price vector for the respective
team member who observes his price only is represented by a bold line. In other words, $I^{h,1} (q)$ is given by the vertical line, while $I^{h,2} (q)$ is given by the horizontal line. Example 1 shows a $\phi$ that is not compatible with equilibrium because the arbitrage opportunity is CK and allows for perfect coordination. Asset demands and supplies will be unbounded and market clearing fails.

**Insert figure 1 here**

We see that it is not necessary that the realization be CK. In this case it is sufficient that it be CK that there is an arbitrage, since it is then also CK which exact strategies are to be implemented, i.e. perfect coordination is possible. Formally, the event $\Omega$ is self-evident\(^6\) and implies the event that $\{ q \in \Omega : q_1 > q_2 \} = \Omega$.

In Figure 2, team member $(h,1)$ knows there is an arbitrage, but $(h,2)$ does not. The bold line represents the set of all possible prices according to member $(h,1)$'s information set, and it is included in the set of arbitrage prices. Since by definition a self-evident set $F$ needs to include $I^{h,a} (q)$ at each $q \in F$, $a = 1, 2$, any candidate $F$ needs to include both the horizontal and vertical lines. But then any candidate $F$ would have a non-empty intersection with $Q$, and therefore could not imply the event $E$ of an arbitrage.

**Insert figure 2 here**

The next example (Figure 3) shows a $\phi$ and a price realization at which both team members know there is an arbitrage, but no-one knows that the other knows.

**Insert figure 3 here**

An example where $\phi$ and the price realization are such that both know there is an arbitrage, member 2 knows that 1 knows it, but 1 does not know that 2 knows is illustrated in Figure 4.

**Insert figure 4 here**

\(^6\)An event $F \in B^A$ is **self-evident** between the subset of traders $\{(h,a)\}_{a \in A'}$ if for all $\omega \in F$ we have $I^{h,a} (\omega) \in B^A$ and $I^{h,a} (\omega) \subseteq F$, all $a \in A' \subseteq A$. It can be shown that an event $E$ is common knowledge between $\{(h,a)\}_{a \in A'}$ in the state $\omega \in \Omega$ iff there is a self-evident event $F \in B^A$ for which $\omega \in F \subseteq E$. 

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Figure 5 illustrates how much larger the set of admissible price functions can be, compared to the standard model, simply because common knowledge is replaced by near-common knowledge. Even at a price realization like the one given by the bold dot it is easy to verify that it is not common knowledge that there is an arbitrage. Thus there is some probability of miscoordination which guarantees that their portfolio problem is well-defined, even in the presence of an arbitrage that is nearly common knowledge.

Insert figure 5 here

It is again the sure gain characteristic of arbitrage that is instrumental. On the previous figures, we assume that $E \geq 2$ and that $\phi$ is well-behaved, by which we mean that $\phi(\mathcal{E})$ is an $A$-dimensional submanifold of $\mathbb{R}^A$. In that case, all that matters is the support of $q$, $\phi(\mathcal{E})$. The actual distribution is irrelevant.

When instead we assume either that $E = 1$, or that $E \geq 2$ but $\phi$ is not well-behaved, then pathological cases may appear. On the next figure the support of prices is one-dimensional and monotonic, thereby any one price fully reveals the other one. It follows that $\phi$ is not admissible.

Insert figure 6 here

The following equilibrium locus L1 is again one-dimensional, but it is nevertheless an admissible pricing function. The admissibility is not a robust characteristic, though, since a small perturbation of L1 to L2 or to L3 generates a locus which is not admissible as it permits the team members to perfectly coordinate their arbitrage portfolios for any realization on the locus below the dotted line on L2 or above the dotted line on L3.

Insert figure 7 here

4.2 Domain of the Pricing Function in terms of CK

In order to summarize the previous discussion a bit more formally, we define $\mathcal{G}$ to be the powerset of $A$, the set of all subsets of $A$. An action protocol for a subset $G \in \mathcal{G}$ of traders of investor $h$ is a vector of Baire functions
\( \alpha^h_G := \{ \alpha^h_{G}^a \}_{a \in G} \in \mathbb{B}^G \), The action protocol is a contingency plan which specifies which action each player executes depending on the price he faces. Within the complement of \( Q \), there are a finite number of basic arbitrage regions, each representing one particular basic form, or category, of arbitrage. For instance, consider

\[
R = \begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

The categories of basic arbitrage opportunities are \( q_1 \leq 0, q_2 \leq 0, q_3 \leq 0, q_3 \geq q_1, q_2 + 2q_3 \leq 2q_1 \) and \( q_2 + q_3 \leq q_1 \). If \( R \) contains redundant assets, some basic opportunities are characterized by strict inequalities rather than by weak ones. A basic category is an event \( E_i, i \in I := \{1, \ldots, k\} \). Those basic categories can then be combined into opportunities \( A_F := \cup_{i \in F} E_i, I' \subset I \). For a given \( G \in \mathcal{G} \), define the set of basic opportunities that lead to unambiguous arbitrage opportunities to the group \( G \) of traders by

\[
Z_G := \{ I' \subset I : \{ \alpha_G \cdot q_G < 0, R_G \alpha_G \geq 0 \} \text{ or } \{ \alpha_G \cdot q_G \leq 0, R_G \alpha_G > 0 \}, \\
\forall q_G \in A_{I'}, \text{ some } \alpha_G \in \mathbb{B}^G \}
\]

The set of states in which such an event \( A_F \cap \phi(\mathcal{E}) \) is CK among a given group \( G \) of investor \( h \) is

\[
E_G^h(\phi) := \{ \epsilon \in \mathcal{E} : A_F \cap \phi(\mathcal{E}) \text{ is CK to } G \text{ at } \epsilon \text{ for some } I' \in Z_G \}
\]

If it is CK among \( G \) at \( \epsilon \) that there is an arbitrage, does that necessarily also mean that they are able to exploit it? The answer is no, as can be seen on Figure 8. In that example assume that there are two assets with identical payoffs. Even though at \( \epsilon \) it is CK to the two traders that there is an arbitrage, they are not able to exploit it. While agent 2 knows the unique arbitrage opportunity with certainty, agent 1 does not know which of the two arbitrage opportunities consistent with his information he is facing. He does therefore not know whether to go long or short, and agent 2 knows that agent 1 cannot take an unambiguous arbitrage action. Depending on the relative probabilities of facing one or the other arbitrage opportunity, the agents may engage in risk-arbitrage only. It follows that the realisation
\( \bar{\epsilon} \not\in E^h_{1,2}(\phi) \). No protocol that agents could have decided upon can lead to a coordinated arbitrage strategy at \( \bar{\epsilon} \). For instance, if agents ex-ante agreed upon the action profile \( \{a^{h,1}(\phi_1(\bar{\epsilon})) = K, a^{h,2}(\phi_2(\bar{\epsilon})) = -K\} \), agent 1 would not know whether the action profile thus specified amounts to an arbitrage at \( \bar{\epsilon} \). Of course, agents could agree ex-ante to not follow the prescriptions of an action profile. For instance, 1 could randomize and choose to go long. Since this action would not be known by agent 2 however, coordination is not feasible.

And even if the exact basic arbitrage opportunity is CK among a group \( G \), they may not be able to exploit it as it may require the intervention of a trader outside of \( G \). In the four states and three assets example at the beginning of this section, assume that \( q_2 + q_3 \leq q_1 \) is CK to \( G = \{1, 2\} \) at some \( \bar{\epsilon} \). \( a^h_G = \{-K, K\} \) is not an arbitrage strategy since come state 4 tomorrow, the strategy loses money. It may be a good deal though if it was also commonly known to \( \{1, 2\} \) that \( q_3 > 0 \). But if \( q_3 \leq 0 \) is a possibility, \( q_1 < q_2 \) cannot be excluded even though \( q_2 + q_3 \leq q_1 \). No action profile leads to a sure gain. If the opportunity was CK to \( G = \{1, 2, 3\} \), perfect coordination would be possible.

As a further example to motivate the specification of \( E^h_{G,G} \), consider again the simple example with two identical assets, with \( \phi \) as on Figure 9. Even though the arbitrage opportunity is CK among the agents at \( \bar{\epsilon} \), there is still an indeterminacy of action protocols. For instance, denoting the extended real numbers by \( \mathbb{R}^* \), \( \{a^{h,1}(\phi_1(\bar{\epsilon})), a^{h,2}(\phi_2(\bar{\epsilon}))\} = \{-K, K\} \), or

\[ \{1 - \frac{(1+K)q_1}{q_1-q_3}, \frac{(1+K)q_1}{q_1-q_3}\} \] are arbitrage portfolios for any \( K \in \mathbb{R}^*_+ \). This does not, however, create a coordination problem, because the traders of investor \( h \) decide, given \( \mathcal{F}^h \), upon one particular action protocol. The same reasoning applies more generally to events of the type \( \cap_{i \in I} E_i \) when more than one basic opportunity is CK. Neither basic arbitrage opportunities nor arbitrage portfolios need to be unique since coordination is possible: come state \( \bar{\epsilon} \), blindly following the protocol yields an arbitrage profit, and this fact is CK. Such arbitrage situations cannot occur at a competitive REE, which is why the pricing function belongs to

\[ \Phi := \{\phi : \mathcal{E} \to \mathbb{R}^A, \mathcal{F}/\mathcal{B}^A\text{-measurable, such that } \cup_{h \in H, G \in \mathcal{G}} E^h_G(\phi) = \emptyset\} \]

As suggested by Proposition 2, both characterizations coincide:

\footnote{The Bayesian equivalent is \( \mu(\cup_{h \in H, G \in \mathcal{G}} E^h_G(\phi)) = 0 \), where the relevant knowledge operator is applied.}
Proposition 3 $\Phi = \bar{\Phi}$

This means that $\phi$ cannot be such that, for realizations in a set of positive measure, the traders of some investor could have devised demand functions ex-ante (that satisfy the asset-specificity and informational requirements imposed by the joint assumptions) that allow them to make arbitrage profits with those action protocols in those states, and for it to be common knowledge between the involved traders that those trades do form an arbitrage portfolio. We repeat that it is, however, not necessary to exclude all pricing functions that allow arbitrage opportunities to be common knowledge, since it would exclude pricing functions that allow for arbitrage prices at which arbitrage is common knowledge between a group $G$ of members, but at which the specific actions to take in order to profit from the opportunity are not commonly known and where coordination is not feasible. Also notice that this formulation eliminates single-agent arbitrages as well, i.e. arbitrages in which an agent $(h, a)$ knows there is an arbitrage that he can exploit on his own. For instance, a bond with a sure payoff of 1 in each state cannot have a non-positive price. In the next session we shall study the investor's optimal action profiles if $\phi \in \Phi$.

5 The Investor Problem

In this section we analyze the optimal investment decisions of a rational investor or household. Such an investor's views of the workings of an economy can be summarized by his forecast function, $\phi$. Assume that $\phi : \mathcal{E} \to \mathbb{R}^A$ is $\mathcal{F}/\mathcal{B}^A$ measurable, where $\mathcal{B}^A$ is the appropriate Borelian sigma-algebra. Since we assume that $\phi \in \Phi$ (here the Bayesian version is sufficient), coordinated arbitrages are excluded, and the Bayesian decision problem of finding optimal schedules involves probabilistic considerations rather than knowledge considerations. Conditional expectations and distributions call for a probabilistic model, and we therefore assume that the decision maker’s information is given by the sigma-algebras $\mathcal{F}^{h,a} = \sigma(e^h, \phi_a)$ rather than by partitions.

The household problem, given $\sigma(e^h)$, consists in choosing Baire action protocols, or demand schedules $f_a^h(q_a, e^h)[\phi]$, one for every asset $a$. A more insightful way of deriving such schedules is to consider the interim problem [P,ii] where we imagine that a particular team member $(h, b)$ observes $q_b$ only
and decides upon the optimal market order to submit:\(^8\)

\[
\max_{\theta^h_b \in \mathbb{R}} E[U^h \| \mathcal{F}^{h,b}]
\]

\[
= \max_{\theta^h_b \in \mathbb{R}} E \left[ u^h_0 \left( u^h_0 - \sum_{a \neq b} f^h_a(\phi_a, e^h)\phi_a - \theta^h_b q_b \right) \right] \quad \text{[P:i]}
\]

\[
+ \sum_s p_s u^h_s \left( u^h_s + \sum_{a \neq b} d_{a,s} f^h_a(\phi_a, e^h) + d_{b,s} \theta^h_b \right) \| \mathcal{F}^{h,b} \]

Assume for ease of interpretation that the differentiation and integration operators commute. Then the first order condition with respect to \(\theta^h_b\) for an interior solution is

\[
q_b = \sum_s d_{b,s} \left[ \frac{E[p_s v^h_s'(u^h_s + \sum_a f^h_a(\phi_a, e^h)d_{a,s})\| \mathcal{F}^{h,b}]}{E[v^h_0'(u^h_0 - \sum_a f^h_a(\phi_a, e^h)q_a)\| \mathcal{F}^{h,b}]} \right]
\]

The term in the outer brackets, call it \(\lambda^{h,b}_s\), is a state-price that is specific to asset \(b\). The intertemporal marginal rate of substitution cannot, in contrast to standard models, play the role of a stochastic pricing kernel since the \(\lambda^{h,b}\) used in the pricing of asset \(b\) does not typically price any other asset. This is the crucial asset-pricing implication of this model that will allow us to generate inefficient and mispriced equilibrium prices without recourse to ad-hoc assumptions. Informational innovations in one part of the markets are not immediately and simultaneously incorporated into all other prices, as standard REE models assume.

No consistent information set can be attributed to investor \(h\). In fact, our model very simply derives different priors for different traders of the same investor endogenously. An econometrician who is not aware of the consequences of our twin market microstructure features would be tempted to...
conclude that the investor violates Bayesian updating and therefore rationality, while in fact the actors in this economy are very rational indeed so as to overcome the impediments inherent in the market microstructure.

Given the demand functions for the other assets, \( f^h_{a_b}(\cdot) \), and the realization \( q_b \), the FOC determines the optimal amount of asset \( b \) to buy, \( \theta^h_b \). Repeating that exercise for every value of \( q_b \), we can trace out a demand function for asset \( b \), depending on the assumed schedules for all other assets. However, we still have to guarantee that there are such functions solving the fixed-point problem in which each demand function depends on all other demand functions:

**Proposition 4 (Demand functions)** Assume the investment opportunity set is truncated to a compact set. For given \((\epsilon^h, \phi)\), and under the standard assumptions, there exists a demand function \( f^h_a \in \mathbb{B} \), with \( f^h_a : q_a \mapsto \theta^h_a = f^h_a(q_a, \epsilon^h)|\phi \), for any asset \( a \in A \).

Compared to a Radner economy, demand functions are defined for a larger set of price functions, which has interesting repercussions for equilibria. In this paper, in view of the well-known difficulties of establishing the existence of equilibria in economies where agents refine their information by the information revealed by the endogenous pricing function (e.g. Green (1977), but also see Allen (1985a)), we shall focus in the next section on some properties of equilibria, assuming an equilibrium exists.

## 6 Equilibrium

The joint assumption of immediacy risk and of asset-specificity leads to a set of admissible pricing functions that could be much richer than the standard one. For one, the support of prices is larger, leading to more dispersion and possibly to more volatility. Prices are less tied down by CK to their “fundamental value” than in standard economies since the investors who would have to bid prices towards their fundamental values themselves face additional uncertainty. It has in fact recently been argued that \( \phi^R(\mathcal{E}) \) is unable to address many asset pricing regularities, and that therefore behavioural biases need to be brought into the modelling. We do not take issue with this, but we merely point out that some of the perceived failures of the standard rational model may lie more in the assumptions of a frictionless trading mechanism than in the rationality criterion itself, however defined.
For instance markets do occasionally exhibit mispricings that either cannot arise in standard models, or that are argued a global unconstrained rational arbitrageur should be able to exploit and therefore to eliminate. Our model may generate arbitrage opportunities in equilibrium with positive probability, depending on \((E, \phi, R)\). The reason that they can be sustained at an equilibrium is of course that the opportunity is not common knowledge to any team of traders. Since in our economy prices are determined locally, \(q_a = d_a \cdot \lambda^a, \ a \in A\), there is no guarantee that there is a \(\lambda \gg 0\) for which \(q_a = \phi_a(\epsilon) = d_a \cdot \lambda, \ \forall a \in A\). As a function of \(\epsilon\), prices of different assets are determined by different state-prices and depend on local information, and there is no reason why the resulting price vector \(q\) should not admit free lunches. This is clearest if we assume that the state of information \(\epsilon_a\) only affects demand for asset \(a\), in which case \(q_a = d_a \cdot \lambda^a(\epsilon_a)\). As long as the fact that \(q\) represents an arbitrage is not common knowledge for any subset of agents \((h, A'), \ A' \subset A\), there is no force driving prices into the set of no-arbitrage prices since each asset price reveals too little about all other asset prices.

For any \(\phi \in \Phi\), the ingredients needed to induce arbitrage opportunities at an equilibrium, in the sense that \(\phi(E) \cap Q \neq \emptyset\), are threefold. First, \(\phi\) must be well-behaved. In particular, \(\phi\) must not map all points \(\epsilon \in E\) into “too small a set,” which must be inside \(Q\). Second, it is obvious that \(E\) needs to be “large enough,” else the equilibria will be too close to the Radner REE (if \(\phi\) is well-behaved), which cannot allow for any arbitrage. Lastly, we have to allow for asset payoffs that are general enough so that an arbitrage can be constructed at all. This latter requirement is guaranteed by the assumption that there is a portfolio \(y\) such that \(Ry > 0\).

If these conditions are satisfied, it becomes apparent that arbitrage is possible, for we have enough degrees of freedom to choose \(\epsilon\) for which \(\{\lambda \in \mathbb{R}^5 : R^\prime \lambda = \phi(\epsilon)\} \cap \mathbb{R}^5_+ = \emptyset\). A simple example (taken from Zigland (2002)) may be instructive.

**Example 1** Consider the CAPM economy (quadratic Von Neumann - Morgenstern utility functions over time two consumption and linear time zero utility function over time one consumption). Assume also that \(\epsilon \in \mathbb{R}^A\) represents a stochastic asset supply, assumed to be multivariate normal \(N(0, \Sigma_\epsilon)\).

Because the support of \(\epsilon\) is all of the real space, \(\mathcal{E} = \mathbb{R}^A\), this specification simplifies the equilibrium existence problem. There cannot be any situation where it is common knowledge to some subset of traders that there is an
arbitrage opportunity, $\mu \left( \cup_{h \in H, G \in G} E^h, G(\phi) \right) = 0$, and the domain for $\phi$ is the set of all $\mathcal{F}/\mathcal{B}^4$- measurable functions.

Then it can be shown that the REE pricing function $\phi$ is an isomorphism, $\phi(\epsilon) = F + G \epsilon$, with parameters $F \in \mathbb{R}^4$ and $G$ a diagonal and invertible $A \times A$ matrix. Arbitrage may therefore occur with strictly positive $\mu-$ probability (a simple proof is in the appendix).

We now turn to the study of the informational (in)efficiency of markets. More precisely, we analyze the information transmission across exchanges or across pits at these equilibria. If each pit is hit by an independent demand or supply shock and if no investors can trade on more than one pit, no information can be transmitted. In our setup, however, prices are correlated so that each observed asset price reveals information about all the states of information. Still, at a given typical realization $(q_0, e^h)$, markets remain inefficient for the team members (of course an econometrician will ex-post be able to see all prices and therefore be able to know more about the state of information), and therefore information is not efficiently aggregated into the price of any given asset. While diverse payoff-relevant factors affect the "true value" of an asset, they may be revealed only in conjunction with the observation of other asset prices. Formally,

**Proposition 5 (Degree of Informational Efficiency.)** Assume that the equilibrium price function $\phi$ is smooth. Define $(h, b)$'s observation function $i^{h,b} : \mathcal{E} \to \mathbb{R}^{2S+P+2}$ by $i^{h,b} : \epsilon \mapsto (\phi_0, e^h)$, a smooth mapping between smooth manifolds. Then for a dense set of prices and individual characteristics $(q_0, e^h)$, whose complement is of Lebesgue measure zero in $\mathbb{R}^{P+1}$, the dimension of residual uncertainty facing team member $(h, b)$, equivalently the degree of inefficient information aggregation, is $(H - 1) (2S + 1 + P) - 1$.

A further issue raised by the recent debates on rationality is the issue of unequal priors, and trading that can be generated by such unequal priors. Our model very simply derives different priors for different traders endogenously. One consequence of this is the fact that trading volume in our model may be much higher than in standard models which suffer from the Milgrom-Stokey syndrome, without having recourse to exogeneously imposing unequal priors.
7 Conclusion

This paper proposes a plausible framework in which to address general equilibrium pricing implications.

First, we show that under standard assumptions, some micro market structure aspects (if taken one at a time) are not crucial because equilibria coincide with standard REE.

Second, we analyze a class of models in which the market structure does matter. We do this by combining two plausible and appealing conditions on how orders are computed, submitted and executed. The implication is that a piece of information on one market needs not be immediately reflected in all other prices. These marketstructure assumptions dispense with noise trading as a means to avoid complete information revelation and thus help avoid the Grossman-Stiglitz paradox (see Grossman and Stiglitz (1980)). While agents would have been perfectly informed from prices in a standard economy, calling for a blurring device in order to generate imperfect information and an incentive to gather and pay for informative signals, no such device is necessary here.

Third, we show that given the imperfect knowledge of any one trader about the fundamentals of the entire economy, equilibria may exhibit miscalculations, and therefore arbitrage opportunities.

The proposed marketstructure assumptions break perfect risk-sharing, enable investors to gain from better information and allow us to view markets much more as the decentralized places they are, as opposed to the idealized frictionless simultaneous worldwide auction envisaged by Walras. The model can be interpreted as generating a trading environment within an institution that is organized along distinct trading desks. Even though the trading desks try to coordinate as much as possible in order to exploit market inefficiencies to a maximum, the market microstructure may not allow a full exploitation. Limits to arbitrage arise from within each investing institution. These limits to efficient markets may be further exacerbated by strategic coordination difficulties across investing institutions, along lines similar to the ones in Morris and Shin (1999).
Appendix: Proofs

Proof of Proposition 1. 1. At a standard REE, investors observe the price realization $q = \phi(\epsilon)$ and solve their asset allocation problem

$$\theta^{h,*} = \arg \max_{\theta \in \mathbb{R}^4} E[U^h||\sigma(e^h, \phi)] \quad [R]$$

Since each optimal portfolio $\theta^{h,*}$ is measurable w.r.t. $\sigma(e^h, \phi)$, this generates a Baire function (Borel-measurable function $f : \mathbb{R}^n \to \mathbb{R}^n$) $f^{h,*}$ satisfying $\theta^{h,*} = f^{h,*}(q, \epsilon^h)$ (refer for instance to Krickeberg (1965), Theorem 2.5 on p.139).

On the other hand, assume the same standard investor optimization problem, but with the added feature of price uncertainty at the time zero optimization stage. Recall that investors have common prior beliefs (summarized by the measure $\mu$) over states of information $\epsilon$. They choose, given their information $\mathcal{F}^h = \sigma(e^h)$, an (version of the) optimal Baire function $f^h : \mathbb{R}^4 \to \mathbb{R}^4$ s.t.

$$f^h = \arg \max_{F : \mathbb{R}^4 \to \mathbb{R}^4} E \left[ u^h_0 \left( w^h_0(\epsilon) - \phi(\epsilon) \cdot F(\phi(\epsilon)) \right) + \sum_s p_s u^h_s \left( d_s \cdot F(\phi(\epsilon)) + w^h_s(\epsilon) \right) \bigg| \sigma(e^h) \right] \quad [IR]$$

First, we show that the argmax $f^h$ of the immediacy risk problem [IR] also solves the Radner problem [R]. Assume to the contrary that, for $q \in W$ where $W \in \sigma(e^h)$ with $\mu(W||\sigma(e^h)) > 0$, $f^h(q) \neq f^{h,*}(q)$. Then on $W$ we have that

$$E[U^h(f^{h,*})||\sigma(e^h, \phi)] > E[U^h(f^h)||\sigma(e^h, \phi)]$$

The function $F$ defined as $F = f^h$ if $q \not\in W$ and $F = f^{h,*}(q)$ if $q \in W$ then satisfies

$$E[U^h(F)||\sigma(e^h)] > E[U^h(f^h)||\sigma(e^h)]$$

contradicting the maximality of $f^h$.

Second, we show that the function $f^{h,*}$ also solves [IR]. Since there is no ex-ante link between the events represented by the sets in $\sigma(e^h, \phi)$, the investor can optimize event by event, and the derived demand functions are the standard Marshallian demand functions. In other words, since $E[U^h||\sigma(e^h)] = E[E[U^h||\sigma(e^h, \phi)]||\sigma(e^h)]$, maximizing $E[U^h||\sigma(e^h, \phi)]$ automatically maximizes the entire expression.
It follows that the resulting equilibrium price vector will indeed be in the set of no-arbitrage prices, which we denote by \( \mathcal{Q} \) (defined in Section 4), and coincides with the REE equilibrium of the economy without immediacy risk (ignoring issues arising from multiple and sunspot equilibria).

2. Since we abstract from immediacy risk, the investor is able to observe the Radner price vector \( q^R = \phi^R(\epsilon) \) before forming his demand, and submits the demand function in the Radner economy, \( f_{a}^{R,h}(q_{a};q_{-a}^{R}) \), as a function of the first variable, \( q_{a} \), only. The investor simply hides the second element \( q_{-a}^{R} \) from the auctioneer. The auctioneer solves \( \sum_h f_{a}^{h}(q_{a};q_{-a}^{R}) = 0 \) for \( q_{a}^{R} = \phi^R(\epsilon) \): the standard equilibrium is trivially unaffected by this constraint. ■

**Proof of Lemma 1** Indeed, an argument identical to Radner’s (Radner (1979)) shows that the following system of aggregate excess demands (a superscript “\( R \)” means the demand function is a standard REE demand function) has typically no solution (the two different realizations of the joint signals are \( \epsilon \) and \( \epsilon' \)):

\[
\begin{align*}
\sum_h f_{a}^{R,h}(q, \epsilon) &= 0 \quad (\forall a \in A) \\
\sum_h f_{a}^{R,h}(q, \epsilon') &= 0 \quad (\forall a \in A) \\
q_{b} &= q_{b}' \\
\epsilon &\neq \epsilon'
\end{align*}
\]

Let us denote the Radner REE by \( \phi^R \), i.e. the Radner equilibrium price which solves \( \sum_h f_{a}^{R,h}(q_{a}, q_{-a}^{R}, \epsilon^{h}) = 0 \), all \( a \in A \). Now assume asset-size specificity away. Then every single price is fully revealing all the other prices via \( q_{-a} = \phi_{-a}^{-1}(q_{a}) \). Denote such a price vector by \( q^{R} \). Now do impose asset-specificity as well, and the investor hands over the schedule

\[
f_{a}^{h}(q_{a}, \epsilon^{h}) = f_{a}^{R,h}(q_{a}, \phi_{-a}^{R}((\phi_{a}^{R})^{-1}(q_{a})), \epsilon^{h})
\]

to the auctioneer clearing market \( a \), and the full-communication equilibrium ensues, as \( q^{R} = \phi^R(\epsilon) \) solves this new system of equations as well: \( \sum_h f_{a}^{R,h}(q_{a}^{R}, \phi_{-a}^{R}((\phi_{a}^{R})^{-1}(q_{a}^{R})), \epsilon^{h}) = 0 \), all \( a \) because \( \phi_{a}^{R}((\phi_{a}^{R})^{-1}(q_{a}^{R})) = q_{-a}^{R} \).

■
Proof of Proposition 3. First, we show that $\Phi \subseteq \bar{\Phi}$. To the contrary of $\phi \in \Phi$, assume that there is some $(h, G, I')$ for which $E^h_G(\phi) = \{q : q \in K_{h,G}^{I'}, \text{ some } I' \subseteq Z_G\} \neq \emptyset$. So it is CK between $G$ at each $q \in K_{h,G}^{I'}$ that there is a strategy $\alpha$:

$$\{\check{\alpha}^h_G(q) \cdot q < 0, \quad R\check{\alpha}^h(q) \leq 0\} \quad \text{or} \quad \{\check{\alpha}^h_G(q) \cdot q \leq 0, \quad R\check{\alpha}^h(q) > 0\}$$

Choose $\alpha^h_G \in \mathcal{H}^G$ by setting $\alpha^h_G(q) := 0$, $\alpha^h_G(q) := 0$ for $q \in \phi(\mathcal{E}) \setminus K_{h,G}^{I'}$ and $\alpha^h_G(q) := \check{\alpha}^h_G(q)$ for $q \in K_{h,G}^{I'}$. We show that $\phi \notin \Phi$. Consider any realization $\epsilon$ for which $\phi(\epsilon) \in K_{h,G}^{I'}$, or alternatively $\phi(\epsilon) \in K_{h,G}^{I'} \neq \emptyset$, in which case $\epsilon \in M(\alpha^h)$. It follows that $\delta \subseteq M(\alpha^h)$ and $\delta \cap M(\alpha^h) \neq \emptyset$, from which we can conclude that $\phi \notin \Phi$.

Second, we show $\Phi \subseteq \bar{\Phi}$. Assume that $\phi \notin \Phi$, i.e. there is a strategy $\bar{\alpha}^h$ s.t. $K_{h}^{I'} \bar{\alpha}^h \neq \emptyset$. Such an $\bar{\alpha}^h$ exploits one or several basic arbitrage categories. It is w.l.o.g. to assume that $\bar{\alpha}^h$ exploits $I' \subseteq Z_G$, for some $G$. Also, for simplicity choose $G$ so that there is no $G' \subseteq G, G' \neq G$, for which $I' \subseteq Z_G$. In other words, $G$ is the smallest group that can unambiguously exploit the basic opportunity. Then define $\alpha^h_G := 0$, $\alpha^h_a(q_a) := 0$ if $q_a \in \phi_a(\mathcal{E}) \setminus \prod_a \mathcal{A}_r$ and $\alpha^h_a(q_a) = \bar{\alpha}^h_a(q_a)$ otherwise, $a \in G$. $\prod_a$ is the projection operator on the $a$th coordinate. So now assume that $q = \phi(\epsilon) \in \mathcal{A}_r$ obtains. Since there is such a strategy $\bar{\alpha}^h$ as outlined above, $(h, a) \in G$ knows that given $q_a$, there is, irrespective of $q_{G \setminus a} \in \phi(\phi^{-1}(q_a))$ an arbitrage of category $I'$, $I_{h,a}^h(q) \subseteq \mathcal{A}_r$. This is true $\forall q \in \mathcal{A}_r$ and $\forall a \in G$, so that $\mathcal{A}_r$ is self-evident between $G$. Therefore $\mathcal{A}_r$ is CK between $G$, and $\phi \notin \Phi$. 

Proof of Proposition 4 (Existence of Demand Functions) The proof goes along similar lines as the proof of Proposition 1. The fixed point problem can be transformed into an easier problem as follows. Rather than maximize the interim objective, given $\mathcal{F}^h_a$, we maximize the ex-ante objective $[P:ea]$, i.e. given $\mathcal{F}^h$, by choosing (for a given $\epsilon^h$) Baire functions $f_a^h : \mathbb{R} \to \mathbb{R}$ for all $a \in A$ (which we could again denote by $f_a^h(q_a, \epsilon^h)|\phi)$:

$$V^h(\epsilon^h)[\phi] = \max \left\{ \left[ f_a^h(q_a, \epsilon^h)|\phi \right] : (F_a^h)_{a=1}^{A} \text{ measurable} \right\}$$

[30]
If this problem admits a solution, this same solution also has to be a solution to each team member’s problem. Indeed, let \( f^* \) represent the solution to the ex-ante problem, and let \( \theta^*_b(q_b) \in \mathbb{R} \) represent the solution to the interim (given \( q_b \)) problem of

\[
\max_{\theta_b^* \in \mathbb{R}} E[U^h|\mathcal{F}^h]\]

Keeping in mind that \( E[U^h|\mathcal{F}^h] = E[ E[U^h|\mathcal{F}^{h,b}]|\mathcal{F}^h] \), assume that \( f^*_b(q_b) \) does not maximize the interim problem (for a nonnull set of prices \( q_b \in A \)). Then setting \( f_b(q_b) = \theta^*_b(q_b) \) for \( q_b \in A \) and \( f_b(q_b) = f^*_b(q_b) \) for \( q_b \in \mathbb{R} \setminus A \) yields a strictly higher ex-ante utility, contradicting the optimality of \( f^* \).

Denote by \( \mathcal{L}^C_{D,1} \) the normed vector space of Borel-measurable functions \( h : \mathbb{R}^C \rightarrow \mathbb{R}^D \) with the norm

\[
\|h\| := E^C[|h|] = E^C \left[ \left( \sum_{i=1}^{D} h_i^2 \right)^{1/2} \right]
\]

using the relevant marginal measure \( \mathbb{P}_C \). Obviously, for certain pricing functions, the optimization problem may not admit a solution unless we restrict the domain of choice. Truncate the investment opportunity set by insisting that investors choose \((f, x) \in K \cap B := (K' \times K'') \cap B, K' \) and \( K'' \) compact and convex sets in \( (\mathcal{L}^1_{1,1})^A \) (by asset-specificity) and \( \mathcal{L}^{S+1}_{A,1} \) respectively, containing zero, and \( B := \{(f, x) : x = w + \left[-\phi/R\right] f\} \) the (convex) budget set.

The constraint set is compact and convex. The demand functions solve the truncated problem:

\[
\max_{(f, x) \in K \cap B} W(x) := E[U^h(x)|\mathcal{F}^h]
\]

The objective function \( W \) is continuous and strictly concave in \( x \). To see continuity, notice that \( |U^h(x_n)| \leq \sup_{x \in K''} U^h(\bar{x}) \leq M_U \), some \( M_U < \infty \). So by the Lebesgue convergence theorem,

\[
\lim_{n \to \infty} E[U^h(x_n)|\mathcal{F}^h] = E[ \lim_{n \to \infty} U^h(x_n)|\mathcal{F}^h] = E[U^h(\lim_{n \to \infty} x_n)|\mathcal{F}^h]
\]

where the last equality follows from the assumed continuity of \( U^h(\cdot) \). Strict concavity is evident: \( W(\lambda x' + (1 - \lambda)x'') = E[U^h(\lambda x' + (1 - \lambda)x'')|\mathcal{F}^h] \)

\[
< E[\lambda U^h(x') + (1 - \lambda)U^h(x'')|\mathcal{F}^h] = \lambda W(x') + (1 - \lambda)W(x'').
\]

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Now since $K \cap B$ is compact and $W$ continuous, there is an argmax $(x^*, f^*)$. By the strict concavity in $x$, $x^*$ is unique, but $f^*$ evidently need not be. 

Proof that Arbitrage can Occur with Positive Probability in Example 1 Consider the nonempty set $Y := \{y \in \mathbb{R}^A : Ry > 0\}$. It is sufficient to show that the set $P^* := \{q \in \mathbb{R}^A : \exists y \in Y : q'y \leq 0\}$ has strictly positive derived measure over asset prices, $\mathbb{P} := \mu \circ \phi^{-1}$. Define the set $P := -Y$. We show first that $P \subset \text{int}(P^*)$, and second that $\mathbb{P}(P) > 0$.

$P$ only contains arbitrage prices: take any $q \in P$ and let the asset demand be $y = -q \in Y$. Then since $y \in Y$, $Ry > 0$, and since $y = -q$, $q'y = -q'q < 0$: $P \subset \text{int}(P^*)$.

Upon inspection, it is clear that the linear mapping defined by $R$ is transversal to $\mathbb{R}_{++}^S$. It follows that $Y$ is a manifold of dimension $A$ and has positive $A$-Lebesgue measure, and so does $P$. Notice that $\phi(\mathcal{E}) \cap P \neq \emptyset$. Since the pricing function $\phi$ is an isomorphism, the preimage of $P$ in $\mathcal{E}$ via $\phi$ has positive $\mu$-measure as well.

Proof of Proposition 5. Similar to Allen (1985b) and Heifetz and Polemarchakis (1998). By Sard’s Theorem (Guillemin and Pollack (1974)), the set of critical values of $i^{h, b}$ is of Lebesgue measure zero and its complement, call it $Z$, is dense. By the regular value theorem, $i^{h, b^{-1}}(q_b, e^h)$ is an $E - 2S - P - 2 = (H - 1)(2S + 1 + P) - 1$ dimensional smooth manifold for every $(q_b, e^h) \in Z$. 

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8 Figures

Figure 1: $\phi$ is not admissible. The arbitrage opportunity is CK and allows for perfect coordination.

Figure 2: $\phi$ is admissible. 1 knows there is an arbitrage, but 2 doesn't since from observing $q_2$ he cannot exclude that $(q_1, q_2)$ does not lie in the upper cone, which is the set of no-arbitrage prices.
Figure 3: $\phi$ is admissible. Both know there is an arbitrage, but no agent knows that the other agent knows.

Figure 4: $\phi$ is admissible. Both know there is an arbitrage, 2 knows that 1 knows, but 1 does not know whether 2 knows.
Figure 5: $\phi$ is admissible. Both know there is an arbitrage, and each one knows that the other one knows, and all know that all know that all know etc., but not ad infinitum.

Figure 6: $\phi$ is not admissible since at any price realization in the set of arbitrage prices, it is common knowledge that the price is an arbitrage price.
Figure 7: Locus L1 is admissible since at any price realization in the set of arbitrage prices, team member \((h, 1)\) does not know whether \((h, 2)\) knows. On Locus L2, however, if the price realization is given by the circle, the arbitrage opportunity is CK. L3 is not admissible since any price realization on the locus above the dotted line represents an arbitrage opportunity that allows the traders to perfectly coordinate.
Figure 8: Both assets have identical payoffs. The equilibrium price locus is given by the bold curve. While at the price realization corresponding to the bold point it is CK between the two agents that there is an arbitrage, agent 2 knows the exact nature of the arbitrage opportunity while agent 1 cannot know whether he should go long or short. $\phi$ is admissible.

Figure 9: Both assets have identical payoffs. The price locus is given by the bold curve. The nature of the arbitrage opportunity at the price realization corresponding to the bold dot is CK and unambiguous to both agents, so perfect coordination is feasible, and $\phi$ is not admissible.