Equilibrium Theory beyond Arbitrage

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Abstract

The notion of arbitrage is predominantly used as a conceptual framework of finance and economics for studying equilibrium as well as pricing relations of asset markets. This may be no longer useful, however, once the transitivity preferences is dropped because equilibrium can exist in an economy which admits arbitrage opportunities. Thus, the no arbitrage conditions are no longer necessary for the existence of equilibrium with non-transitive preferences. We propose a new condition which is necessary and sufficient for the existence of equilibrium of the economy where preferences need not be transitive and the consumption set of each agent need not be bounded from below.

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I. Introduction

The consumption set need not be bounded from below in an asset market economy where unlimited short sales are allowed. One important example is the capital asset pricing model developed by Sharpe (1964), Lintner (1965) and Markowitz (1952). The existence of a Walrasian equilibrium with unbounded-from-below choice sets was initially addressed in Hart (1974) who introduced a condition on preferences eventually known as a no arbitrage condition. To generalize the condition of Hart (1974) on preferences, different arbitrage notions have been introduced in the literature (see for example, Hammond (1983), Page (1987), Werner (1987), Chichilnisky (1995), Dana et al. (1999), Allouch (2002), among others). The arbitrage conditions are not only sufficient but also necessary for the existence of a Walrasian equilibrium in certain cases.

It is important to note that all the equilibrium existence results with unbounded consumption sets make use of the assumption of transitivity of preferences. However, once the transitivity of preferences is dropped, we show by means of example that a Walrasian equilibrium can exist without satisfying any of the no arbitrage conditions found in the literature. Hence, without transitivity arbitrage opportunities may exist in Walrasian equilibrium. Indeed, we will demonstrate that without transitivity of preferences the standard arbitrage conditions are no longer necessary for the existence of a Walrasian equilibrium.¹

Also in the finance literature, it has been documented that arbitrage opportunities may exist in equilibrium. This literature explains that market frictions such as short-selling constraints, transaction costs, margin requirements etc. prevent agents from exhausting away arbitrage opportunities.² Our example illustrates, however, that arbitrage opportunities survive frictionless markets if preferences of agents are not transitive. Moreover, it is widely recognized in the literature that transitive preferences fail to explain important anomalous phenomena such as the equity premium puzzle and the preference reversal phenomenon. For example, experimental methods, Grether and Plott (1979) and Loomes et al. (1991) among others document the violation of transitivity of preferences by detecting the preference reversal phenomenon.

¹Experimental work of Berg et al. (1985) documents that arbitrage profits can be extracted from the optimal choices which reveal the preference reversal phenomenon. Preference reversals arise when an agent prefers lottery A to lottery B but sets a higher selling price on B than on A. Such behavior violates transitivity of preferences.
The purpose of this paper is to prove the existence of a Walrasian equilibrium with unbounded consumption sets and non-transitive preferences. To this end we introduce a new condition which is necessary and sufficient for the existence of a Walrasian equilibrium. For any sequence of individually rational and feasible allocations, this condition enables us to find a convergent sequence of approximately feasible allocations whose preferred sets are supportable at the original allocations with the prices which supports the original preferred sets. Here the supporting prices need not be nonzero. If the original sequence of allocations are out of equilibrium in general for some well-chosen economies with compactified consumption sets, then the supporting prices turns out to be zero and therefore, the price support for the alternative sequence of allocations is trivial. Otherwise, the supporting prices will be nonzero. If the original sequence of allocations is in equilibrium for the compactified economies and it is bounded, then its limit point turns out to be in equilibrium for the economy. If it is unbounded, then its bounded alternative together with the nonzero supporting prices turns out to converge to an equilibrium of the economy.

This new condition turns out to subsume as a special case all the arbitrage conditions found in the literature. In particular, we construct an example of an economy with non-transitive preferences in which the set of Pareto optimal allocations is compact, a Walrasian equilibrium exists, but any known arbitrage conditions are violated. In other words, the economy of the example has all the desired properties except for transitivity of preferences but the existence of equilibrium cannot be explained by any conditions found in the existing literature. That is, the arbitrage conditions are no longer necessary. Furthermore, our analysis covers interdependent preferences. Thus, the equilibrium existence result of the paper includes as a special case not only all the equilibrium existence results with unbounded consumption sets but also gives as a corollary the standard equilibrium existence results without transitivity or completeness of preferences.

The paper makes the following advances in the literature.

First, we prove a new abstract equilibrium result a la Shafer-Sonnenschein which is based on weaker continuity conditions on preferences than those of Shafer and Sonnenschein (1975), and Yannelis and Prabhakar (1983). This result does not follow from the results of Shafer and Sonnenschein (1975), and Yannelis and Prabhakar (1983).

Second, we apply the above equilibrium results to an exchange economy with arbitrary consumption sets where agents’ preferences may be interdependent and need not be tran-
sitive or complete or even have an open graph. The application of this result to an exchange economy enables us to obtain a new Walrasian equilibrium existence theorem which gives as a corollary the classical existence results. This preparatory and preliminary result allows for arbitrary consumption sets but the set of individually rational and feasible allocations is assumed to be bounded. Despite the bound on the set of individually rational and feasible allocations, this result does not follow from any of the equilibrium existence results with unbounded consumption sets, e.g. Hart (1974), Hammond (1983), Werner (1987), Page (1987), Dana et al (1999), Allouch (2002), among others. This equilibrium result should be viewed as an auxiliary theorem to obtain the main result of the paper.

Third, by means of the new condition which is introduced in this paper, we prove our main existence theorem for Walrasian equilibrium with unbounded consumption sets. We also indicate how this new condition is necessary for the existence of a Walrasian equilibrium. It should be emphasized that the sufficient and necessary condition for the equilibrium existence does not imply that the utility set of individually rational allocations is closed. (In contrast, the arbitrage conditions introduced by Hart (1974), Hammond(1983), Werner (1987), Page (1987), Dana et al. (1999), Allouch (2002), among others lead to the compactness of the utility set. One exception is Won (2001) which shows the existence of equilibrium in the economy with the possibly non-compact utility set.)

II. Abstract Economies

2.1 Notation

\( \mathbb{R} \) denotes the set of real numbers.
\( \mathbb{R}_+ \) denotes the set of nonnegative real numbers.
\( \mathbb{R}_{++} \) denotes the set of strictly positive real numbers.
\( \mathbb{R}^l \) denotes the \( l \)-fold Cartesian product of \( \mathbb{R} \).
\( \|x\| \) denotes the norm of the vector \( \|x\| \).
\( x^n \rightarrow x \) denotes the convergence of the sequence \( \{x^n\} \) to the point \( x \).
\( 2^A \) denotes the set of all nonempty subsets of the \( A \).
\( co \, A \) denotes the cone generated by the set \( A \).
con A denotes the convex hull of the set A.
closure A denotes the closure of the set A.
int A denotes the interior of the set A.
∂A denotes the boundary of the set A.

B(x, r) denotes the open ball in \( \mathbb{R}^l \) centered at the point x with radius \( r > 0 \).
∅ denotes the empty set.

2.2 Definitions

For two nonempty subsets Z and Y in \( \mathbb{R}^l \), consider a correspondence \( \varphi : Z \to 2^Y \). Let \( cl \varphi \), \( int \varphi \) and \( con \varphi \) denote the correspondence from Z to \( 2^Y \) which has the value \( cl \varphi(z) \), \( int \varphi(z) \) and \( con \varphi(z) \) for all \( z \in Z \), respectively. The correspondence \( \varphi \) is said to have an open graph if \( G_{\varphi} \equiv \{(z, y) \in Z \times Y : y \in \varphi(z)\} \) is open in \( Z \times Y \). The correspondence \( \varphi \) is said to have open lower sections if the set \( \varphi^{-1}(y) = \{z \in Z : y \in \varphi(z)\} \) is open in Z for every \( y \in Y \) and \( \varphi \) is said to have open upper sections (or open-valued) if \( \varphi(z) \) is open in Y for every \( z \in Z \). The correspondence \( \varphi \) is said to be lower semi-continuous if for every open set \( V \) of \( Y \), \( \{z \in Z : \varphi(z) \cap V \neq \emptyset\} \) is open in Z and \( \varphi \) is said to be upper semi-continuous if for every open set \( V \) of \( Y \), \( \{x \in X : \varphi(x) \subset V\} \) is open in \( X \).

2.3 Equilibrium in Abstract Economies

Let I denote the set of agents. Assume that I is a countable set. For each \( i \in I \), let \( X_i \) be a nonempty set in \( \mathbb{R}^l \). We set \( X = \prod_{i \in I} X_i \). An abstract economy \( \Gamma = \{(X_i, A_i, P_i) : i \in I\} \) is a set of ordered triples \( (X_i, A_i, P_i) \) where \( A_i : X \to 2^{X_i} \) and \( P_i : X \to 2^{X_i} \) are correspondences. The abstract economy provides a simple but powerful conceptual framework for studying an exchange economy in a general setting.

**Definition 2.3.1**: An equilibrium for \( \Gamma \) is a point \( x \in X \) such that for all \( i \in I \),

(i) \( x_i \in cl A_i(x) \)
(ii) \( P_i(x) \cap cl A_i(x) = \emptyset \).

We are now ready to provide the following preliminary theorem which will be useful in proving the existence of equilibrium of an exchange economy.
Theorem 2.3.1: Let $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$ be an abstract economy satisfying the following conditions for each $i \in I$

A1. Each $X_i$ is convex, compact and nonempty in $\mathbb{R}^l$.  
A2. Each $P_i$ is lower semi-continuous and has open upper sections.  
A3. $A_i$ is convex-, nonempty-valued and has an open graph.  
A4. $clA_i$ is upper semi-continuous.  
A5. $x_i \notin con P_i(x)$ for all $x \in X$.

Then $\Gamma$ has an equilibrium, i.e. there exists $x^* \in X$ such that for all $i \in I$,

(i) $x^*_i \in clA_i(x^*)$, and
(ii) $P_i(x^*) \cap clA_i(x^*) = \emptyset$.

Proof: For each $i \in I$, define $\psi_i : X \to 2^{X_i}$ by $\psi_i(x) = con P_i(x) \cap A_i(x)$. Clearly, $\psi_i$ is convex-valued. For each $i \in I$, let $U_i = \{x \in X : \psi_i(x) \neq \emptyset\}$. Since $P_i$ is lower semi-continuous, by Proposition 2.6 in Michael (1956) $con P_i$ is lower semi-continuous. Hence by Lemma 4.2 of Yannelis (1987), $\psi_i$ is lower semi-continuous.\footnote{Let $T$ and $Y$ be any topological spaces, and $\theta_1 : T \to 2^Y$ and $\theta_2 : T \to 2^Y$ be correspondences. Then Lemma 4.2 of Yannelis (1987) shows that if $\theta_1$ has an open graph and $\theta_2$ is lower semi-continuous, the correspondence $\hat{\theta} : T \to 2^Y$ defined by $\hat{\theta}(t) = \theta_1(t) \cap \theta_2(t)$ is lower semi-continuous.\footnote{X is metrizable because it is a countable product of metric spaces. It is also well-known (Stone’s Theorem) that metrizable spaces are paracompact.}} It follows from the lower semi-continuity of $\psi_i$ that for each $i \in I$, $U_i$ is open in $X$ (recall that $U_i = \{x \in X : \psi_i(x) \cap X \neq \emptyset\}$).

There are two cases to be examined; for all $i \in I$, $U_i$ is either (a) empty or (b) nonempty. By the condition A2, $P_i$ is open-valued. Thus it is easily seen in case (a) that for all $i$ and for all $x \in X$, $\psi_i(x) = con P_i(x) \cap A_i(x) = \emptyset$ and therefore, $P_i(x) \cap A_i(x) = \emptyset$ for every $x \in X$. Recalling that $P_i$ is open-valued, we conclude that for all $i$ and for all $x \in X$, $P_i(x) \cap clA_i(x) = \emptyset$. Hence condition (ii) of the theorem holds. To show that (i) is also fulfilled, we define the correspondence $A : X \to 2^X$ by $A(x) = \prod_{i \in I} clA_i(x)$. Since each $clA_i$ is upper semi-continuous, closed, convex and nonempty-valued, so is $A$. By the Kakutani fixed point theorem there exists $x^* \in X$ such that $x^* \in A(x^*)$, which implies that $x^*_i \in clA_i(x^*)$ for all $i \in I$. Thus (i) also holds.

We now turn to case (b). Since $U_i$ is open in $X$, it is also paracompact.\footnote{X is metrizable because it is a countable product of metric spaces. It is also well-known (Stone’s Theorem) that metrizable spaces are paracompact.} Denote by $\psi_i|_{U_i}$ the restriction of $\psi_i$ to $U_i$, i.e., $\psi_i|_{U_i} : U_i \to 2^{X_i}$. By applying Theorem 3.1′′′ of Michael (1956, pp368) to $\psi_i|_{U_i}$, there exists a continuous function $f_i : U_i \to X_i$ such that $f_i(x) \in \psi_i(x)$ for
all \( x \in U_i \). For each \( i \in I \), define \( F_i : X \to 2^{X_i} \) by

\[
F_i(x) = \begin{cases} 
\{ f_i(x) \}, & \text{if } x \in U_i, \\
\text{cl} A_i(x), & \text{if } x \notin U_i.
\end{cases}
\]

By Lemma 6.1 in Yannelis and Prabhakar (1983), \( F_i \) is upper semi-continuous and it is clearly convex-, nonempty- and closed- valued. Define \( F : X \to 2^X \) by \( F(x) = \prod_{i \in I} F_i(x) \). Then \( F \) is upper semi-continuous, convex, nonempty, and closed-valued. By the Kakutani fixed point theorem, there exists \( x^* \in X \) such that \( x^* \in F(x^*) \), i.e. \( x^*_i \in F_i(x^*) \) for all \( i \in I \). If \( x^* \in U_i \) for some \( i \in I \), then \( x^*_i = f_i(x^*) \in \text{con} P_i(x^*) \cap A_i(x^*) \subseteq \text{con} P_i(x^*) \) which contradicts A5.

Hence for all \( i \in I \), \( x^*_i \notin U_i \), i.e. \( x^*_i \in \text{cl} A_i(x^*) \) and \( \text{con} P_i(x^*) \cap A_i(x^*) = \emptyset \), which implies \( P_i(x^*) \cap \text{cl} A_i(x^*) = \emptyset \). Since \( P_i \) is open-valued, we can conclude that \( P_i(x^*) \cap \text{cl} A_i(x^*) = \emptyset \), i.e., \( x^* \) is an equilibrium for \( \Gamma \).

**Remark 2.3.1:** Theorem 2.3.1 doesn’t follow from Shafer and Sonnenschein (1975) or Yannelis and Prabhakar (1983) because the assumptions on the correspondences \( P_i \) are weaker here than those papers. In particular, Yannelis and Prabhakar (1983) assume that \( P_i \) must have open lower sections which implies that \( P_i \) is lower semi-continuous but the reverse is not true. Shafer and Sonnenschein (1975) assume that preference correspondence must have an open graph which implies that both sections (upper and lower) must be open.

**Remark 2.3.2:** It is worth mentioning that Theorem 2.3.1 holds in separable Banach spaces. The reason is that Michael selection theorem for lower semi-continuous correspondences still holds for separable Banach spaces provided that the correspondence has a non-empty norm interior. The reader can easily see that the correspondence \( \psi \) is open-valued. However, for non-separable Banach spaces the proof of Theorem 2.3.1 fails and a different continuous selection theorem is required, (see Yannelis and Prabhakar (1983)).

**Remark 2.3.3:** The assumptions on the constraint correspondences, however, are slightly stronger than those of Shafer and Sonnenschein (1975). Nonetheless they are automatically fulfilled by the standard exchange economy, and as we will see in the next section Theorem 2.3.1 will enable us to provide a more general Walrasian equilibrium existence result than that of Shafer (1976).

Gale and Mas-Collel (1975) consider the economy \( \tilde{\Gamma} = (X_i, A_i, \tilde{P}_i)_{i \in I} \) where \( \tilde{P}_i \) is defined...
as follows; for all \( x_i \in X_i \),
\[
\tilde{P}_i(x) = \{(1 - \alpha)x_i + \alpha x'_i : 0 < \alpha \leq 1, x'_i \in \text{con} P_i(x)\}.
\]
For each \( x \in X \), \( \tilde{P}_i(x) \) is convex. For each \( i \in I \), we consider the following condition.

**C5.** For all \( x \in X \) with \( P_i(x) \neq \emptyset \), \( P_i(x) \) is convex, \( x_i \notin P_i(x) \), and \( x_i \) is in the boundary of \( P_i(x) \).

This condition is stronger than A5 but the following result shows that instead of A5, C5 can be used together with the other assumptions to prove Theorem 2.3.1.

**Proposition 2.3.1:** If \( \Gamma \) satisfies the assumptions A1-A5, then \( \tilde{\Gamma} \) satisfies the assumptions A1-A4 and C5. Moreover, if \( x \in X \) is an equilibrium of \( \tilde{\Gamma} \), it is also an equilibrium of \( \Gamma \).

**Proof:** Suppose that \( \Gamma \) satisfies the assumptions A1-A5. Since \( P_i \) is lower semi-continuous, by Proposition 2.6 in Michael (1956) \( \text{con} P_i \) is lower semi-continuous and therefore, \( \tilde{P}_i \) is lower semi-continuous (for details, see Gale and Mas-Colell (1975) or Allouch (2002)). Let \( x \) be a point in \( X \) with \( P_i(x) \neq \emptyset \). By A5 we immediately see \( x_i \notin \tilde{P}_i(x) \) and by construction, \( x_i \) is in the boundary of \( \tilde{P}_i(x) \).

Now we show that A2 allows \( \tilde{P}_i \) to have open upper sections. Suppose that there exists \( x \in X \) with \( P_i(x) \neq \emptyset \) such that \( \tilde{P}_i(x) \) is not open. Then there exists \( z \in \tilde{P}_i(x) \) such that for any \( r > 0, B(z, r) \cap X_i \notin \tilde{P}_i(x) \). We can pick \( x'_i \in \text{con} P_i(x) \) and \( \alpha \in (0, 1] \) such that \( z = (1 - \alpha)x_i + \alpha x'_i \). Since \( \text{con} P_i(x) \) is open in \( X_i \), there exists \( r' > 0 \) which satisfies \( B(x'_i, r') \cap X_i \subset \text{con} P_i(x) \). By definition,
\[
B(z, \alpha r') = \{(1 - \alpha)x_i\} + \alpha(B(x'_i, r') \cap X_i) \subset \tilde{P}_i(x).
\]
It gives \( B(z, \alpha r') \cap X_i \subset \tilde{P}_i(x) \), which is impossible. Thus \( \tilde{P}_i \) has open upper sections. We conclude that each \( \tilde{P}_i \) satisfies A2 and C5, and thus \( \tilde{\Gamma} \) satisfies A1-A4 and C5.

Suppose that \( x \in X \) is an equilibrium of \( \tilde{\Gamma} \). Then \( x \in \text{cl} A_i(x) \) and \( \tilde{P}_i(x) \cap \text{cl} A_i(x) = \emptyset \) for all \( i \in I \). Since \( P_i(x) \subset \tilde{P}_i(x) \), we trivially see that \( x \in \text{cl} A_i(x) \) and \( P_i(x) \cap \text{cl} A_i(x) = \emptyset \) for each \( i \). Thus \( x \) is also an equilibrium of \( \Gamma \).

\[\square\]

**2.4 A Generalization of the Classical Walrasian Equilibrium Existence Theorem**
An exchange economy is considered which is populated with finitely many agents in $I$. We let $I$ denote both the number and the set of agents. For each $i \in I$, let $e_i \in \mathbb{R}^l$ denote the initial endowment and $X_i \subset \mathbb{R}^l$ denote the choice set of agent $i \in I$. We denote the exchange economy by $E = \{(X_i,e_i,P_i) : i \in I\}$ where $P_i : X \rightarrow 2^{X_i}$ is a correspondence. For points $x \in X$ and $y_i \in X_i$, we read $y_i \in P_i(x)$ as “agent $i$ strictly prefers $y_i$ to $x_i$ provided that the other agents choose $x_j$ for all $j \neq i$.” For example, $P_i$ can represent the preference ordering $\succ_i$ on $X$. In this case, for any vector $x = (x_1, \ldots, x_I) \in X$ the set $P_i(x)$ denotes the upper contour set $\{y_i \in X_i : (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_I) \succ_i x\}$ of $\succ_i$ at $x$. The preference ordering $\succ_i$ on $X$ is so general that it allows interdependence among agents and need not be either transitive or complete. For each $p \in \mathbb{R}^l \setminus \{0\}$ and each $i \in I$, we define the set $B_i(p) = \{x_i \in X_i : p \cdot x_i < p \cdot e_i\}$. Clearly, $clB_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$ whenever $B_i(p) \neq \emptyset$ for all $p \in \mathbb{R}^l \setminus \{0\}$.

**Definition 2.4.1:** An equilibrium for the exchange economy $E$ is a pair $(p,x) \in (\mathbb{R}^l \setminus \{0\}) \times X$ such that

(i) $x_i \in clB_i(p)$ for all $i \in I$,

(ii) $P_i(x) \cap clB_i(p) = \emptyset$ for all $i \in I$, and

(iii) $\sum_{i \in I}(x_i - e_i) = 0$.

Let $F_1$ denote the set of feasible allocations $\{x \in X : \sum_{i \in I}(x_i - e_i) = 0\}$. For each $x \in X$ and each $i \in I$, let $R_i(x_i) = \{z \in X : x_i \notin con P_i(z)\}$ and $F_2(x) = \cap_{i \in I}R_i(x_i)$. The set $F_1$ consists of feasible allocations. An allocation $x \in F_2(e)$ is called individually rational. Such individual rationality is appropriate in the sense that if $x$ is an equilibrium allocation, then $e_i \notin con P_i(x)$ for all $i \in I$.

We set $\bar{F} = F_1 \cap F_2(e)$. Then $x \in \bar{F}$ is an allocation which is feasible and individually rational. In the special case that the preference ordering of agent $i$ is defined on $X_i$ and representable with a quasiconcave utility function $u_i$ for all $i \in I$, $F_2(e)$ is equal to the set $\{x \in X : u_i(e_i) \leq u_i(x_i)\}$ for all $i \in I$ and therefore, it is convex. We assume that $E$ satisfies the following conditions for all $i \in I$.

**B1.** $X_i$ is a closed, nonempty and convex set in $\mathbb{R}^l$.

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5In the case that $\succ_i$ is defined on $X_i$ for all $i$, one might be tempted to define the set of individually rational allocations as $\prod_{i \in I}\{x_i \in X_i : x_i \succeq_i e_i\}$. This is fine as far as $\succ_i$ is transitive for all $i \in I$. It is not the case, however, with non-transitive preferences because $x_i \succ_i e_i$ does not imply $P_i(x_i) \subset P_i(e_i)$, as illustrated in Example 2.4.1.
**B2.** \(e_i\) is in the interior of \(X_i\).\(^6\)

**B3.** \(P_i\) is lower semi-continuous and has open upper sections.

**B4.** For all \(x \in X\), \(x_i \notin con P_i(x)\).

**B5.** For all \(x \in \tilde{F}\), \(P_i(x) \neq \emptyset\).

Assumption B5 states that no satiation occurs on the set of feasible and individually rational allocations. Define \(\varphi_i : X \rightarrow 2^{X_i}\) by \(\varphi_i(x) = con P_i(x)\). For \(e_i \in X_i\), notice that

\[
\varphi_i^{-1}(e_i) = \{x \in X : e_i \in \varphi_i(x)\} = \{x \in X : e_i \in con P_i(x)\}.
\]

Observe that

\[
X \setminus \varphi_i^{-1}(e_i) = \{x \in X : e_i \notin \varphi_i(x)\} = \{x \in X : e_i \notin con P_i(x)\} = R_i(e_i).
\]

Since \(P_i\) is lower semi-continuous under the condition B3, \(\varphi_i^{-1}(e_i)\) need not be open and therefore, \(\tilde{F}\) need not be closed.\(^7\) Below we indicate what are the conditions which guarantee that for each \(i \in I\) and each \(e_i \in X_i\), \(R_i(e_i)\) is closed.

**Lemma 2.4.1:** For each \(i \in I\), let \(P_i : X \rightarrow 2^{X_i}\) be a preference correspondence with open lower sections, i.e., for each \(y_i \in X_i\) the set \(P_i^{-1}(y_i) = \{x \in X : y_i \in P_i(x)\}\) is open in \(X\). Then for each \(i \in I\) and \(e_i \in X_i\), the set \(R_i(e_i) = \{z \in X : e_i \notin con P_i(z)\}\) is closed in \(X\).

**Proof:** For all \(i \in I\), define \(\varphi_i : X \rightarrow 2^{X_i}\) by \(\varphi_i(x) = con P_i(x)\). Then by Lemma 5.1 of Yannelis and Prabhakar (1983), \(\varphi_i\) has open lower sections, i.e. for each \(y_i \in X_i\), \(\varphi_i^{-1}(y_i) = \{x \in X : y_i \in \varphi_i(x)\}\) is open in \(X\). Since \(R_i(e_i) = X \setminus \varphi_i^{-1}(e_i)\), we conclude that for each \(e_i \in X_i\), \(R_i(e_i)\) is closed in \(X\). \(\square\)

**Remark 2.4.1:** Notice that if \(P_i : X \rightarrow 2^{X_i}\) has open lower sections, it is also lower semi-continuous. However, the converse is not true (see Yannelis and Prabhakar (1983)). In this

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\(^6\)With additional complications, this condition can be relaxed such that \(e_i\) is in the relative interior of \(X_i\).

\(^7\)If each \(P_i\) has open lower sections, then \(\tilde{F}(e)\) is closed. But this is not warranted by B3. For example, the budget correspondence is lower semi-continuous but does not have open lower sections. See Yannelis and Prabhakar (1983) for an example of a lower semi-continuous correspondence which doesn’t have open lower sections.
paper, preferences are assumed to be lower semi-continuous. Thus for each \( i \in I \), \( R_i(e_i) \) need not be closed.

Let \( F \) denote the closure of \( \tilde{F} \) in \( X \). By B4, \( e_i \notin \text{con} P_i(e) \) for all \( i \in I \) and therefore, \( e \in F_2(e) \). Since \( e \in F_1 \), it follows that \( e \in F \). In particular, \( F \) is not empty. For each \( i \in I \), let \( X_i^F \) denote the projection of \( F \) onto \( X_i \).

If \( \tilde{F} \) is not bounded, no equilibrium may exist. One can construct an economy which satisfies B1-B5 but admits no Pareto optimal allocation and therefore, no equilibrium simply because \( \tilde{F} \) is unbounded.

**Example 2.4.1:** (The following example is not new. But some detailed discussion follows because its slight modification will be useful in the sequel.) Consider a one-period economy with two agents and one good in each state of nature. We assume that one of the two events 1 and 2 occurs in the end of the period but the true state is not revealed until then. Two Arrow securities are traded in the beginning of the period which pay in the end of the period. Let \( v_j \) denote the \( j \)th asset holdings for each \( j = 1, 2 \). We assume that no short-selling restrictions are imposed on selecting portfolio, i.e. \( X_1 = X_2 = \mathbb{R}^2 \).

We suppose that agent 1 believes with probability one that event 2 will occur in the end of the period while agent 2 believes with probability one that event 1 will occur. We assume that preferences are defined on \( \mathbb{R}^2 \) and follow the expected utility hypothesis. Then for a point \( v = (v_1, v_2) \in \mathbb{R}^2 \), preferences of both agents can be represented by \( u_1(v) = v_2 \) and \( u_2(v) = v_1 \). Clearly, \( F = \tilde{F} \) and \( F \) is unbounded.

We show that this economy has no equilibrium. Let \( (v, v') \) be an allocation of portfolios in \( F \). Then \( u_1(v) = v_2 \) and \( u_2(v') = v'_1 \). In particular, we see that for any \( \lambda > 0 \),

\[
\begin{align*}
u_1((v_1, v_2) + \lambda (-1, 1)) &= v_2 + \lambda > u_1(v_1, v_2) \\
u_2((v'_1, v'_2) + \lambda (1, -1)) &= v'_1 + \lambda > u_2(v'_1, v'_2) \\
\{(v_1, v_2) + \lambda (-1, 1), (v'_1, v'_2) + \lambda (1, -1)\} &\in F
\end{align*}
\]

That is, the utility of agent 1 goes to infinity as his asset holdings increase from \( v \) in the direction of \((-1, 1)\) while the utility of agent 2 goes to infinity as his asset holdings increase from \( v' \) in the direction of \((1, -1)\).\footnote{This implies that \((-1, 1)\) is the direction of recession of the upper contour sets of \( u_1 \) while \((1, -1)\) is the direction of recession of the upper contour sets of \( u_2 \). For details on the direction of recession, see Rockafellar (1970).} Clearly, one of the two portfolios \( \lambda (-1, 1) \) and \( \lambda (1, -1) \)
always has non-positive value at any asset prices in \( \mathbb{R}^2 \). Thus there always exists a free lunch which one of the two agents can enjoy. For example, suppose that \((-1,1)\) has non-positive market value. Then it provides an arbitrage opportunity for agent 1 in that he can increase his utility limitlessly without paying nothing. Thus the economy has no equilibrium. Moreover \{\((−1, 1), (1, −1)\)\} reallocates allocations in \( F \) in a Pareto-improving manner and therefore, there exist no optimal allocations of the economy.

For a set \( Z \) in \( \mathbb{R}^l \), let \( C(Z) \) denote its recession cone. Then the boundedness of \( \tilde{F} \) is characterized as follows.

**Lemma 2.4.2:** Suppose that each \( R_i(e_i) \) is closed. Then \( \tilde{F} \) is bounded if and only if
\[
C(F) \bigcap \bigcap_{i \in I} C(R_i(e_i)) = \{0\}.
\]

**Proof:** We set \( H = \{v \in \mathbb{R}^l : \sum_{i \in I} v_i = 0\} \). It is easy to verify that \( C(F_1) = H \).

(\(\Rightarrow\)) Suppose that there exists a nonzero vector \( w \in H \bigcap \bigcap_{i \in I} C(R_i(e_i)) \). Let \( x \) be a point in \( \tilde{F} \) and \( \lambda \) an arbitrary positive number. Then by definition, \( x + \lambda w \in F_1 \) and \( x + \lambda w \in cl R_i(e_i) \) for all \( i \). Thus for any \( \lambda > 0 \), \( x + \lambda w \in F_1 \bigcap \bigcap_{i \in I} R_i(e_i) = \tilde{F} \). This implies that \( \tilde{F} \) is unbounded.

(\(\Leftarrow\)) Suppose that \( \tilde{F} \) is unbounded. Then there exists \( \{x^n\} \) in \( \tilde{F} \) such that \( \|x^n\| \to \infty \). Let \( v^n = x^n / \|x^n\| \). Since \( \{v^n\} \) is bounded, it converges to some point \( v \in \mathbb{R}^l \). Clearly \( v \neq 0 \). Since \( x^n \) is in \( F_1 \) and in \( R_i(e_i) \) for all \( i \in I \), by Theorem 8.2 of Rockafellar (1970, pp 63) \( v \) is in \( H \) and in every \( C(R_i(e_i)) \). Thus \( v \in H \bigcap \bigcap_{i \in I} C(R_i(e_i)) \). \( \square \)

**Remark 2.4.2:** If \( P_i \) has open lower sections for each \( i \in I \), then by Lemma 2.4.1 the set \( R_i(e_i) \) is closed in \( X_i \). In particular, for each \( i \in I \) the set \( R_i(e_i) \) is closed and convex in the case where preferences are defined on \( X_i \) and are representable by a continuous, quasiconcave utility function on \( X_i \) for each \( i \in I \). In this case, by Lemma 2.4.2, \( \tilde{F} \) is bounded if and only if \( \{C(R_i(e_i))\} \) is positively semi-independent. If no half line is contained in indifference curves, the no arbitrage conditions introduced by Werner (1987) and Page (1987) are equivalent to the positive semi-independence of \( \{C(R_i(e_i))\} \) and therefore, they imply the boundedness of \( \tilde{F} \).

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9The recession cone of a convex set consists of the directions of recession. For details, see Rockafellar (1970).

10For details on the relation between the compactness of \( \tilde{F} \) and the no arbitrage conditions, see Dana, Le Van and Magnien (1999) or Page, Wooders and Monteiro (2000).
We introduce the sets $C$ and $C_1$ in $\mathbb{R}^l$ defined by
\[
C = \{ p \in \mathbb{R}^l : \|p\| \leq 1 \} \\
C_1 = \{ p \in \mathbb{R}^l : \|p\| = 1 \}.
\]

Below we prove a very general Walrasian equilibrium existence theorem.

**Theorem 2.4.1:** Suppose that $E$ satisfies the assumptions B1-B5. Then there exists an equilibrium $(p,x) \in C_1 \times X$ of the economy $E$ if $\tilde{F}$ is bounded.

**Proof:** By Proposition 2.3.1, without loss of generality we may assume that $E$ satisfies B1-B3, B5 and instead of B4, the following condition.

**B4-1.** For all $x \in X$, $P_i(x)$ is convex, $x_i \notin P_i(x)$ and $x_i$ is in the boundary of $P_i(x)$.

Since $F$ is bounded, so is $X_i^F$ for all $i \in I$. Thus we can choose a closed and bounded ball $K$ centered at the origin in $\mathbb{R}^l$ which contains $X_i^F$ and $e_i$ in its interior for all $i \in I$. We introduce the truncated economy $\hat{E} = (\hat{X}_i,e_i,\hat{P}_i)$ where for all $i \in I$,
\[
\hat{X}_i = X_i \cap K, \quad \hat{X} = \prod_{i \in I} \hat{X}_i \quad \text{and} \quad \hat{P}_i(x) = P_i(x) \cap K \quad \text{for all } x \in \hat{X}.
\]

**Step 1:** To apply the result of Theorem 2.3.1, we need to convert $\hat{E}$ into the abstract economy $\Gamma = (\hat{X}_i,A_i,G_i)_{i \in I'}$ where $I' = I \cup \{0\}$ by adding the agent 0 as follows; if $i = 0$, we set $\hat{X}_0 = C$ and define
\[
G_0(p,x) = \{ q \in C : q \cdot (\sum_{i \in I}(x_i - e_i)) > p \cdot (\sum_{i \in I}(x_i - e_i)) \} \\
A_0(p,x) = C \quad \text{for all } (p,x) \in C \times \hat{X}
\]
and if $i \in I$, for all $(p,x) \in C \times \hat{X}$ we set
\[
G_i(p,x) = \hat{P}_i(x) \\
A_i(p,x) = \{ x_i \in X_i : p \cdot x_i < p \cdot e_i + 1 - \|p\| \} \cap K.
\]

Since $e_i$ is in the interior of both $X_i$ and $K$, $A_i(p,x)$ is not empty for all $(p,x) \in C \times \hat{X}$. On the other hand, $cl A_i : C \times \hat{X} \rightarrow 2^K$ has a closed graph and $K$ is compact. These imply that the correspondence $cl A_i$ is upper semi-continuous. Thus $\Gamma$ satisfies A1-A4 and C5. B4-1.

By Theorem 2.3.1, $\Gamma$ has an equilibrium, i.e., there exists $(\hat{p},\hat{x}) \in C \times \hat{X}$ such that
\[
(i) \quad \hat{p} \in G_0(\hat{p},\hat{x}) \quad \text{and} \quad G_0(\hat{p},\hat{x}) \cap C = \emptyset
\]
and for all $i \in I$,

(ii) $\hat{x}_i \in cl A_i(\hat{p}, \hat{x})$ and $G_i(\hat{p}, \hat{x}) \cap cl A_i(\hat{p}, \hat{x}) = \emptyset$.

We will show that (i) and (ii) imply that $(\hat{p}, \hat{x})$ is an equilibrium for the exchange economy $E$.

**Step 2:** We show that $\hat{x} \in F_1$, i.e. $\sum_{i \in I} (\hat{x}_i - e_i) = 0$. Since $\hat{p} \in cl A_0(\hat{p}, \hat{x})$ and $G_0(\hat{p}, \hat{x}) \cap C = \emptyset$, we see that $\|\hat{p}\| \leq 1$ and for all $q \in C$,

$$\hat{p} \cdot (\sum_{i \in I} (\hat{x}_i - e_i)) \geq q \cdot (\sum_{i \in I} (\hat{x}_i - e_i)).$$

Suppose that $\hat{x} \notin F_1$. We set $q' = (\sum_{i \in I} (\hat{x}_i - e_i))/\|\sum_{i \in I} (\hat{x}_i - e_i)\|$. It follows that $q' \in C$ and therefore,

$$\hat{p} \cdot (\sum_{i \in I} (\hat{x}_i - e_i)) \geq q' \cdot \sum_{i \in I} (\hat{x}_i - e_i) = \| \sum_{i \in I} (\hat{x}_i - e_i) \| > 0.$$

On the other hand, $\hat{x}_i \in cl A_i(\hat{p}, \hat{x})$ for each $i \in I$ implies

$$\hat{p} \cdot (\hat{x}_i - e_i) \leq 1 - \| \hat{p} \| \leq 0.$$

Summing up over $i \in I$, we obtain $\hat{p} \cdot (\sum_{i \in I} (\hat{x}_i - e_i)) \leq 0$, which is impossible. Therefore, $\hat{x}$ is in $F_1$.

**Step 3:** We claim that $\hat{x} \in \bar{F}_2(e)$ for any $e \in X$. Suppose otherwise, i.e., $e_i \in P_i(\hat{x})$ for some $i \in I$. Since $e_i \in int K$, there exists $\alpha \in (0, 1)$ such that $\alpha \hat{x}_i + (1 - \alpha)e_i \in K$. By B4-1, we also have $\alpha \hat{x}_i + (1 - \alpha)e_i \in P_i(\hat{x})$ and therefore, $\alpha \hat{x}_i + (1 - \alpha)e_i \in \hat{P}_i(\hat{x})$. Recalling that $\hat{x}_i$ is in $cl A_i(\hat{p}, \hat{x})$, we see that

$$\hat{p} \cdot (\alpha \hat{x}_i + (1 - \alpha)e_i) = \alpha \hat{p} \cdot \hat{x}_i + (1 - \alpha) \hat{p} \cdot e_i \leq \alpha (\hat{p} \cdot e_i + 1 - \| \hat{p} \|) + (1 - \alpha) \hat{p} \cdot e_i \leq \hat{p} \cdot e_i + 1 - \| \hat{p} \|.$$

Hence, $\alpha \hat{x}_i + (1 - \alpha)e_i \in cl A_i(\hat{p}, \hat{x})$ and therefore, $\alpha \hat{x}_i + (1 - \alpha)e_i \in \hat{P}_i(\hat{x}) \cap cl A_i(\hat{p}, \hat{x})$, which is impossible.

**Step 4:** The results of Step 2 and 3 implies $\hat{x} \in \bar{F}$. By B5, we have $P_i(\hat{x}) \neq \emptyset$ for all $i \in I$. We want to show that

$$\| \hat{p} \| = 1 \text{ and } \hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i \text{ for all } i \in I.$$
By B4-1, \( \hat{x}_i \) is in the boundary of the convex set \( \hat{P}_i(\hat{x}) \). Let \( t_i \) be a point in \( \hat{P}_i(\hat{x}) \). Then for any \( \alpha \in (0, 1] \), \( \alpha t_i + (1 - \alpha)\hat{x}_i \) is in \( \hat{P}_i(\hat{x}) \). Since \( \hat{P}_i(\hat{x}) \cap cl A_i(\hat{p}, \hat{x}) = \emptyset \), it implies that for all \( \alpha \in (0, 1] \),

\[
\hat{p} \cdot (\alpha t_i + (1 - \alpha)\hat{x}_i) > \hat{p} \cdot e_i + 1 - \|\hat{p}\|.
\]

By letting \( \alpha \to 0 \), we have \( \hat{p} \cdot \hat{x}_i \geq \hat{p} \cdot e_i + 1 - \|\hat{p}\| \). On the other hand, \( \hat{x}_i \in cl A_i(\hat{p}, \hat{x}) \) implies that \( \hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + 1 - \|\hat{p}\| \). Hence, for all \( i \in I \),

\[
\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + 1 - \|\hat{p}\|.
\]

Summing it over \( I \), we see that

\[
\sum_{i \in I} \hat{p} \cdot (\hat{x}_i - e_i) = \sum_{i \in I} (1 - \|\hat{p}\|).
\]

Since \( \sum_{i \in I} (\hat{x}_i - e_i) = 0 \), we obtain \( \|\hat{p}\| = 1 \). Moreover, we can conclude that \( \hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i \) for all \( i \in I \).

**Step 5:** What is proved up to Step 4 is that \((\hat{p}, \hat{x})\) is an equilibrium for the truncated economy \( \check{E} \). Now we show that \((\hat{p}, \hat{x})\) is an equilibrium of \( E \) by verifying that \( P_l(\hat{x}) \cap cl B_l(\hat{p}) = \emptyset \) for all \( i \in I \). By Step 3 and Step 4, \( \hat{x} \) is in \( F \). Since \( X^F_l \subset int K \), it implies that \( \hat{x}_i \in int K \) for all \( i \in I \). Suppose for some \( i \in I \), there exists \( z_i \in P_l(\hat{x}) \cap cl B_l(\hat{p}) \). Then we can choose \( \alpha' \in (0, 1) \) such that \( \alpha' \hat{x}_i + (1 - \alpha')z_i \in K \). It follows from B4-1 that \( \alpha' \hat{x}_i + (1 - \alpha')z_i \in \hat{P}_i(\hat{x}) \).

Recalling that \( \|\hat{p}\| = 1 \), \( \hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i \) and \( \hat{p} \cdot z_i \leq \hat{p} \cdot e_i \), we have \( \alpha' \hat{x}_i + (1 - \alpha')z_i \leq \hat{p} \cdot e_i = \hat{p} \cdot e_i + 1 - \|\hat{p}\| \), and therefore, \( \alpha' \hat{x}_i + (1 - \alpha')z_i \in \hat{P}_i(\hat{x}) \cap cl A_i(\hat{p}, \hat{x}) \), which is impossible. Consequently, \((\hat{p}, \hat{x}) \in C_1 \times X \) is an equilibrium for \( E \).

The following corollary is a generalization of the existence result of Shafer (1976).

**Corollary 2.4.1:** Suppose that \( E \) satisfies B2-B5 and the following condition.

**B1’**. \( X_i \) is a bounded from below, convex and nonempty subset of \( \mathbb{R}^l \) for all \( i \in I \).

Then there exists an equilibrium \((p, x) \in C_1 \times X \) of the economy \( E \).

**Proof:** By Lemma 2.4.2, B1’ implies that \( \check{F} \) is bounded. Therefore the conclusion follows from Theorem 2.4.1.

**Remark 2.4.3:** In Theorem 2.4.1 transitivity of preferences is not required, and even if preferences are representable by a utility function, the weak continuity assumptions don’t imply
that the set $F_2(e)$ of individually rational allocations is closed. Hence, none of the results which rely on the closedness of $F_2(e)$ can be used to obtain Theorem 2.4.1. It is worth pointing out that the very general existence result of Allouch (2002) is based on the fact that preferences have open lower sections, which implies that $F_2(e)$ is closed.

**Remark 2.4.4:** Corollary 2.4.1 generalizes all classical equilibrium existence theorems for exchange economies because our continuity assumptions are weaker than those of Shafer (1976). In particular,Shafer (1976) assumes that preferences have an open graph, which implies B3 but the reverse is not true. Also our assumptions on the preference correspondences are slightly weaker than those of Gale and Mas-Colell (1974) and moreover, preferences are allowed to be interdependent.

**Remark 2.4.5:** Notice that Theorem 2.4.1 doesn’t follow from the main existence theorem of Dana et al. (1999) or Allouch (Theorem 3.1, 2002). The reason is that preferences are allowed to be interdependent and non-transitive, which are not the case with Dana et al. (1999) or Allouch (2002). However, the condition of $\tilde{F}$ being bounded implies the compactness with partial preorder (CPP) condition of Allouch (2002). In the next section, we will demonstrate how Theorem 2.4.1 can be used to obtain a much more general result than that of Dana et al. (1999) and Allouch (2002).

**III. Main Results**

The notion of arbitrage is very useful in addressing the equilibrium existence issue with the asset market economy where agents have transitive preferences. In particular, no arbitrage opportunity is found in the economy which has an equilibrium. As illustrated below, however, this is not the case with non-transitive preferences because equilibrium can exist in the economy which admits an arbitrage opportunity. An economy is illustrated where an arbitrage opportunity coexists with equilibrium only because of the non-transitivity of preferences. A new condition for the existence of equilibrium is presented which subsumes the no arbitrage conditions as a special case and also covers the counterexample to the no arbitrage conditions. We demonstrate that the condition is necessary and sufficient for the existence of equilibrium.
3.1 Counterexample to the Literature

We will illustrate an example where the existence of equilibrium is not explained by the literature. The example is a slight modification of Example 2.4.1 but its implications to the existence of equilibrium are drastic. Remarkably, an arbitrage opportunity coexists with equilibrium in the economy which shares the desired properties with the literature except for the non-transitivity of preferences. In particular, agents can enjoy an arbitrage opportunity if their choices are slightly perturbed from equilibrium.

For illustrative purposes, we consider the case where preferences for agent \( i \) are represented by a reflexive ordering \( \succeq_i \) on \( X_i \) for all \( i \in I \). Let \( \succ_i \) denote the strict ordering on \( X_i \) induced by \( \succeq_i \); for all \( z_i, x_i \in X_i \), \( z_i \succ_i x_i \) if \( z_i \succeq_i x_i \) but not \( x_i \succeq_i z_i \). As \( \succeq_i \) is transitive, so is \( \succ_i \). Define the correspondence \( Q_i : X_i \to 2^{X_i} \) by \( Q_i(x_i) = \{ z_i \in X_i : z_i \succ_i x_i \} \). In this section, we assume that for all \( x \in X \) and all \( i \in I \), \( P_i(x) = Q_i(x_i) \). For each \( x_i \in X_i \), we set \( \overline{Q}_i(x_i) = \{ z_i \in X_i : z_i \succeq_i x_i \} \). Then it is easy to see that for each \( x \in X \), \( F_2(x) = \prod_{i \in I} \overline{Q}_i(x_i) \).

For a nonempty convex set \( S \subset \mathbb{R}^I \), let \( L(S) \) denote the lineality space of \( S \), the maximal subspace contained in the recession cone \( C(S) \) of \( S \).

**NUA:** Assume that preferences of agent \( i \) are represented by a quasiconcave function \( u_i : X_i \to \mathbb{R} \) for each \( i \in I \) such that \( C(Q_i(e_i)) = C(Q_i(x_i)) \) for all \( x_i \in X_i \). Then \( E \) admits no unbounded arbitrage (NUA) if \( v_i \in C(Q_i(e_i)) \) for all \( i \in I \) and \( \sum_{i \in I} v_i = 0 \) implies that \( v_i \in L(Q_i(e_i)) \) for all \( i \in I \).

**NAO:** Assume that preferences of agent \( i \) are represented by a quasiconcave function \( u_i : X_i \to \mathbb{R} \) for each \( i \in I \) such that \( C(Q_i(e_i)) = C(Q_i(x_i)) \) for all \( x_i \in X_i \) and \( C(Q_i(e_i)) \setminus L(Q_i(e_i)) \neq \emptyset \) where \( C(S) \) denotes the recession cone of the set \( S \). Then \( E \) admits no arbitrage opportunity (NAO) if it satisfies

\[
\bigcap_{i \in I} \{ p \in \mathbb{R}^I : p \cdot v > 0 \text{ for all } v \in C(Q_i(e_i)) \setminus L(Q_i(e_i)) \} \neq \emptyset.
\]

The no unbounded arbitrage (NUA) condition used here is a slight generalization of the original one introduced by Hart (1974) and Page (1987). The no arbitrage (NAO) condition

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11It is implicitly assumed here that \( P_i \) is convex-valued. By Proposition 2.3.1, the convexity assumption does no harm to equilibrium analysis under consideration.

12For details, see Rockafellar (1974)
of Definition 3.1.2 is used in Werner (1987). If \( C(cl P_i(x)) = C(R_i(e_i)) \) for all \( x \in X \), then the NUA condition is also equivalent to the NAO condition of Werner (1987).\(^{13}\) Dana et al. (1999) and Page et al. (2000) generalize the NUA and NAO conditions of Werner (1987) and Page (1987) to the case where \( \tilde{F} \) need not be bounded, and they compare various notions of arbitrage in an exhaustive manner. In particular, the condition of Dana et al. (1999) is equivalent to the compactness of the utility set for the allocations in \( \tilde{F} \). Allouch (2002) further generalizes the notion of arbitrage to the case where preferences are transitive but need not be complete.

**CPP.** (i) The preference ordering \( \geq_i \) on \( X_i \) is transitive and reflexive for each \( i \in I \). (ii) For each sequence \( \{x^n\} \) in \( \tilde{F} \), there exist a subsequence \( \{x^{n_k}\} \) and a sequence \( \{y^{n_k}\} \) in \( X \) convergent to some point \( y \in F \) which satisfies \( y^{n_k}_i >_i x^{n_k}_i \) for all \( n_k \) and for all \( i \in I \).

Allouch (2002) shows that the CPP condition subsumes the arbitrage conditions of Dana et al. (1999) and Page et al. (2000) as a special case. In particular, the CPP condition holds if the utility set is compact. For later comparison, we provide another generalization of arbitrage to the case that preferences need not be transitive.

**Definition 3.1.1:** The economy \( E \) admits no local arbitrage at \( x \in F_2(e) \) if for all \( z \in \prod_{i \in I} Q_i(x_i) \), \( v_i \in C(cl \bar{Q}_i(z_i)) \) for all \( i \in I \) and \( \sum_{i \in I} v_i = 0 \) implies that \( x_i + v_i \in cl \bar{Q}_i(x_i) \setminus Q_i(x_i) \) for all \( i \in I \).

No local arbitrage is admitted at all \( x \in F_2(e) \) if \( E \) satisfies the NUA condition. Suppose that \( E \) satisfies the NUA condition where \( \geq_i \) is represented by a quasiconcave function \( u_i : X_i \to \mathbb{R} \) for each \( i \in I \) such that \( C(\bar{Q}_i(e_i)) = C(\bar{Q}_i(x_i)) \) for all \( x_i \in X_i \). Let \( z \) be a point in \( \prod_{i \in I} Q_i(x_i) \) and \( v_i \) a point in \( C(cl \bar{Q}_i(z_i)) \) for all \( i \in I \) which satisfies \( \sum_{i \in I} v_i = 0 \). By the NUA condition, each \( v_i \) is in \( L(\bar{Q}_i(e_i)) \). Since \( C(\bar{Q}_i(e_i)) = C(\bar{Q}_i(z_i)) \) for any \( z_i \in X_i \), \( v_i \) is in \( L(\bar{Q}_i(z_i)) \). In particular, \( v_i \in C(\bar{Q}_i(x_i)) \) and \( -v_i \in L(\bar{Q}_i(x_i + v_i)) \subset C(\bar{Q}_i(x_i + v_i)) \). The former implies \( u_i(x_i + v_i) \geq u_i(x_i) \) while the latter implies that \( u_i(x_i) = u_i((x_i + v_i) - v_i) \geq u_i(x_i + v_i) \). That is, \( u_i(x_i + v_i) = u_i(x_i) \). Therefore, \( x_i + v_i \in cl \bar{Q}_i(x_i) \setminus Q_i(x_i) \) for all \( i \in I \). Thus the no local arbitrage condition is a generalization of the conditions of Page (1987) and Werner (1987). In particular, preferences need not be transitive, and the condition is defined in a

\(^{13}\)For details, see Page et al. (2000).
point-wise manner.

The following result shows that if the economy has an equilibrium \((p, x)\) with transitive preferences, no local arbitrage is admitted at any allocations in \(\prod_{i \in I} Q_i(x_i)\). That is, the no local arbitrage condition is necessary for equilibrium.

**Lemma 3.1.1** Suppose that \(\succeq_i\) on \(X_i\) is transitive and reflexive for all \(i \in I\). If \(E\) has an equilibrium \((p, x)\), then the no local arbitrage condition holds at \(x\).

**Proof:** Suppose that \(x\) does not satisfy the no local arbitrage condition. Then there exist \(z \in \prod_{i \in I} Q_i(x_i)\) and \(\{v_i\}\) such that \(v_i \in C(cl \overline{Q}_i(z_i))\) for all \(i \in I\), \(\sum_{i \in I} v_i = 0\) and \(x_h + v_h \notin cl \overline{Q}_h(x_h) \setminus Q_h(x_h)\) for some \(h \in I\).

First we show that \(p \cdot v_i = 0\) for all \(i \in I\). Let \(y_i\) be a point in \(X_i\) which satisfies \(y_i \succeq_i z_i\). Since \(z_i \succ_i x_i\), by transitivity we have \(y_i \succ_i x_i\) and therefore, \(\overline{Q}_i(z_i) \subset Q_i(x_i)\). Thus we see that \(C(cl \overline{Q}_i(z_i)) \subset C(cl Q_i(x_i))\), which implies that \(v_i \in C(cl Q_i(x_i))\) for all \(i \in I\). Since it implies \(x_i + v_i \in cl Q_i(x_i)\), there exists \(\epsilon^n \rightarrow 0\) in \(\mathbb{R}^l\) such that \(x_i + v_i + \epsilon^n \in Q_i(x_i)\) for all \(n\). It follows from the equilibrium condition that \(p \cdot e_i < p \cdot (x_i + v_i + \epsilon^n)\) for all \(n\). Passing to the limit, we have \(p \cdot e_i \leq p \cdot (x_i + v_i)\) for all \(i\). Summing it over \(i \in I\), we see that \(p \cdot \sum_{i \in I} e_i \leq p \cdot \sum_{i \in I} (x_i + v_i)\). Since \(x \in F_1\), we must have \(p \cdot \sum_{i \in I} v_i \geq 0\). Recalling that \(\sum_{i \in I} v_i = 0\), it implies that \(p \cdot v_i = 0\) for all \(i \in I\).

We claim that \(x_h + v_h \in Q_h(x_h)\). Since \(\succeq_i\) is reflexive, \(x_h\) is in \(\overline{Q}_h(x_h)\) and therefore, \(x_h \in cl \overline{Q}_h(x_h)\). On the other hand, \(C(cl Q_h(x_h)) \subset C(cl \overline{Q}_h(x_h))\) and therefore, \(v_h \in C(cl \overline{Q}_h(x_h))\). Thus, we have \(x_h + v_h \in cl \overline{Q}_h(x_h)\). Since \(x_h + v_h \notin cl \overline{Q}_h(x_h) \setminus Q_h(x_h)\), it implies that \(x_h + v_h \in Q_h(x_h)\). Recalling that \(p \cdot v_h = 0\), we see that \(x_h + v_h \in cl B_h(p)\) and therefore, \(x_h + v_h \in Q_h(x_h) \cap cl B(p) \neq \emptyset\), which is impossible. \(\square\)

Lemma 3.1.1 is false if transitivity is dropped as indicated in the following the example. In other words, equilibrium can exist without satisfying either the no arbitrage conditions or the CPP condition. Specifically, the coexistence between equilibrium and arbitrage opportunities is possible without transitivity.

**Example 3.1.1:** The economy is the same as in Example 2.4.1 except that preferences are not transitive at the origin of \(\mathbb{R}^2\). We set \(e_1 = e_2 = (0, 0) \in \mathbb{R}^2\). For all \(v = (v_1, v_2) \in \mathbb{R}^2\),
preference are defined by the sets

\[
Q_1(v) = \begin{cases} 
R_{++}^2 & \text{if } v = (0,0) \\
\{v' = (v_1', v_2') \in \mathbb{R}^2 : v_2' > v_2\} & \text{if } v \neq (0,0)
\end{cases}
\]

\[
Q_2(v) = \begin{cases} 
R_{++}^2 & \text{if } v = (0,0) \\
\{v' = (v_1', v_2') \in \mathbb{R}^2 : v_1' > v_1\} & \text{if } v \neq (0,0).
\end{cases}
\]

Preferences of agent \(i\) are the same as in Example 3.1.1 except at \((0,0)\). As shown in Figure 1, \((-1,2) \in Q_1((1,1))\) and \((1,1) \in Q_1((0,0))\) but \((-1,2) \notin Q_1((0,0))\). Thus \(Q_1\) is not transitive. Similarly, we can show that \(Q_2\) is not transitive.

\(<\text{Insert Figure 1}>\)

It is easy to see that the sets \(Q_1\) and \(Q_2\) is written as

\[
\overline{Q}_1 = \{v = (v_1, v_2) \in \mathbb{R}^2 : v_2 \geq 0\}
\]

\[
\overline{Q}_2 = \{v = (v_1, v_2) \in \mathbb{R}^2 : v_1 \geq 0\}.
\]

We show that the economy satisfies all the conditions imposed by B1-B5. Clearly, \(Q_1\) and \(Q_2\) are convex and open valued. We claim that they are lower semi-continuous. For a point \(v \in \mathbb{R}^2\), let \(z = (z_1, z_2)\) be a point in \(Q_1(v)\). Let \(v^n \rightarrow v\) in \(\mathbb{R}^2\). Suppose that \(v = (0,0)\). Then we see that \(z_1 > 0\) and \(z_2 > 0\). Since \(v^n \rightarrow (0,0)\), \(z_2 > v_2^n\) for sufficiently large \(n\) and therefore, \(z \in \mathbb{R}^2_{++} \cap \{v = (v_1, v_2) \in \mathbb{R}^2 : v_2 > v_2^n\}\). It implies that \(z \in Q_1(v^n)\).

Suppose that \(v \neq (0,0)\). Since \(R^2_{++} \setminus \{(0,0)\}\) is open, \(v^n\) is in \(\mathbb{R}^2 \setminus \{(0,0)\}\) for sufficiently large \(n\). On the other hand, \(z_2 > v_2^n\) so that we have \(z \in Q_1(v^n)\) for sufficiently large \(n\). Therefore, we conclude that \(Q_1\) is lower semi-continuous. By the same argument we can show that \(Q_2\) is lower semi-continuous. Thus, we see that the economy satisfies B1-B5.

Recalling that \(Q_1(e_1) = Q_2(e_1) = \mathbb{R}^2_{++}\), we see that any nonzero \(p \in \mathbb{R}^2_{++}\) together with the allocation \((e_1, e_2) \in \bar{F}\) constitutes an equilibrium. However, the no local arbitrage condition is violated at all the allocations in \(\prod_{i \in I} Q_i(e_i)\). Let \(z_i\) be a point that is strictly preferred to \(e_i\), i.e., \(z_i \in Q_i(e_i)\) for each \(i \in I\). We set \(v_1 = (-1, 1)\) and \(v_2 = (1, -1)\). Clearly, \(v_i \in C(Q_i(z_i))\) for all \(i \in I\) and \(v_1 + v_2 = 0\). Moreover, \(z_i + v_i \in Q_i(z_i)\) for each \(i \in I\). Consequently the no local arbitrage condition is violated at the equilibrium allocation \((e_1, e_2)\). This result is quite intuitive. Clearly, one of the two portfolios \((-1,1)\) and \((1,-1)\) always has non-positive value at any asset prices in \(\mathbb{R}^2\). These observations imply that there always exists a free lunch which one of the two agents can enjoy at the allocation \((z_1, z_2)\). For example, suppose
that \((-1, 1)\) has non-positive market value. Then it provides an arbitrage opportunity for agent 1 in that he can improve on his welfare from the position \(z_1\) without any limit and without any cost. As shown in Lemma 3.1.1, however, the coexistence between equilibrium and arbitrages opportunities is impossible with transitive preferences.

Now show that the CPP condition is violated. For each \(n\), set \(x^n_1 = (-a^n, b^n)\) and \(x^n_2 = (a^n, -b^n)\) for some \(a^n\) and \(b^n\) in \(\mathbb{R}\). We assume that \(x^n = (x^n_1, x^n_2)\) is in \(\tilde{F}\) for all \(n\). Then we have \(a^n \geq 0\) and \(b^n \geq 0\). Suppose that \(a_n\) and \(b_n\) are strictly positive and increasing for all \(n\), and \(a^n \to \infty\) and \(b^n \to \infty\). Then the distance of \(Q_i(x^n_i)\) from the origin goes to infinity as \(n \to \infty\). Thus, there is no bounded sequence \(\{y^n\}\) which satisfies \(y^n_i \in Q_i(x^n_i)\) for all \(n\) and all \(i = 1, 2\).

It is interesting to see that arbitrage opportunities undergo an abrupt breakdown at the equilibrium choice due to the non-transitivity of preferences. Let \(\epsilon^n \to 0\) be a sequence in \(\mathbb{R}\) which satisfies \(\epsilon^n > 0\) for all \(n\). Clearly, \((\epsilon^n, \epsilon^n) \in Q_i(e_i)\) and therefore, \((\epsilon^n, \epsilon^n) \in \overline{Q}_i(e_i)\) for each \(i = 1, 2\). Since one of the bundles \((-1, 1)\) and \((1, -1)\) has no positive value at any prices in \(\mathbb{R}^2\), there always exists an agent who can enjoy a free lunch at \((\epsilon^n, \epsilon^n)\) for all \(n\) created by the arbitrage opportunity. The free lunch, however, suddenly disappears at the limit point \((0, 0)\) of \((\epsilon^n, \epsilon^n)\) because preferences fail to be transitive at \((0, 0)\). The present example shows that if preferences are not transitive, then equilibrium can exist in the economy which admits an arbitrage opportunity. This is in sharp contrast to the case with transitive preferences where the existence of equilibrium implies the absence of local arbitrages as shown in Lemma 3.1.1. Hence the existence results in the literature may not be used to prove the existence of equilibrium without transitivity.

Remark 3.1.1: The economy of Example 3.1.1 has the desired properties in terms of preferences except for the non-transitivity. Moreover, the set of allocations in \(\tilde{F}\) that are not Pareto dominated consists of a single point, i.e., it is compact. This property is comparable to the compactness of the utility set in the case where preferences are numerically representable.

Nonetheless, there is no literature which covers Example 3.1.1. The classical works do not apply simply because consumption sets have no lower bound. The line of research following the seminal work of Hart (1974) is not applicable because preferences are not transitive. Most importantly, the existence of equilibrium in Example 3.1.1 cannot be explained either
by the no arbitrage conditions of Werner (1987) and Page (1987) or by the CPP condition of Allouch (2002). By Lemma 3.1.1, the economy of Example 3.1.1 violates the no local arbitrage condition and therefore, the other extensions of the no arbitrage conditions including Dana et al. (1999) and Page et al. (2000) are also violated.

3.2 New Conditions without Transitivity

Example 3.1.1 illustrates that arbitrage-related conditions may not be useful for the existence of equilibrium with non-transitive preferences. We will provide new conditions which subsume the no arbitrage conditions as a special case and are relevant to the case with non-transitive preferences. They turn out to be either sufficient or necessary for the existence of equilibrium. Before going further, it should be mentioned that due to Proposition 2.3.1, without any loss of generality we can assume B4-1 instead of B4 in proving the existence of equilibrium for the economy which satisfies B3 and B4. In particular, we will assume that for any \( z \in X \), \( P_i(z) \) is convex and \( z_i \) is in \( \partial P_i(z) \).

For a point \( z \in X \), we set \( r(z) = \max\{\|z_i\|, i \in I\} \), i.e. \( r(z) \) denote the maximum length of \( z_i \)'s. Let \( x \) and \( y \) be a point in \( X \). We set

\[
V(x) = \{v \in \mathbb{R}^l : \|v\| < r(x)\}
\]

\[
V_1(y) = \{v \in \mathbb{R}^l : \|v\| < r(x) + 1\}
\]

\[
V(x; y) = \{v \in \mathbb{R}^l : \|v\| < \max\{r(x), r(y) + 1\}\}.
\]

The set \( V(x) \) is an open ball whose radius is equal to the maximum length of \( x_i \)'s, \( V_1(y) \) an open ball whose radius is equal to the maximum length of \( y_i \)'s plus one, and \( V(x; y) \) an open ball whose radius is equal to the maximum of the previous two radii. Since \( I \) is finite, there exists \( i \in I \) such that \( x_i \in \partial V(x) \). Clearly, there exists no \( i \in I \) such that \( y_i \in \partial V_1(y) \). By definition, \( V(x) \subset V(x; y) \) and \( V_1(y) \subset V(x; y) \). For each \( i \in I \), let \( L_i(x_i) \) the subspace spanned by the vector \( x_i - e_i \). We set

\[
G(x; y) = \sum_{i \in I} \text{co}( (cl P_i(x) \cap cl V(x; y)) - \{x_i\} ) + \sum_{i \in I} L_i(x_i).
\]

Since \( x_i \in \partial P_i(x) \), \( x_i \in cl P_i(x) \cap cl V(x; y) \) and therefore, \( 0 \in (cl P_i(x) \cap cl V(x; y)) - \{x_i\} \) for all \( i \in I \). Thus \( G(x; y) \) is not empty.

The set \( G(x; y) \) looks complicated but its implications are simple and intuitive.

**Lemma 3.2.1:** For points \( x, y \) in \( X \), \( G(x; y) \neq \mathbb{R}^l \) if and only if there exists a nonzero vector
\( p \in \mathbb{R}^l \) such that \( p \cdot z \geq 0 \) for all \( z \in G(x; y) \), i.e., \( p \cdot x_i \leq p \cdot z_i \) for all \( z_i \in \text{cl} P_i(x) \cap \text{cl} V(x; y) \) and all \( i \in I \), and \( p \cdot x_i = p \cdot e_i \) for all \( i \in I \).

**Proof:** Suppose that \( G(x; y) \neq \mathbb{R}^l \). Since \( G(x; y) \) is a convex cone, by the separating hyperplane theorem there exists a nonzero \( p \in \mathbb{R}^l \) such that for all \( z \in G(x; y) \), \( 0 \leq p \cdot z \).

Recalling that \( 0 \in (\text{cl} P_i(x) \cap \text{cl} V(x; y)) - \{x_i\} \) and \( L_i(x_i) \) for all \( i \in I \), we see that \( \text{co}((\text{cl} P_j(x) \cap \text{cl} V(x; y)) - \{x_j\}) \subset G(x; y) \) and \( L_j(x_j) \subset G(x; y) \) for all \( j \in I \).

Let \( z_j \in \text{cl} P_j(x) \cap \text{cl} V(x; y) \). Then there exist \( \lambda \geq 0 \) and \( z_j' \in \text{co}((\text{cl} P_j(x) \cap \text{cl} V(x; y)) - \{x_j\}) \) such that \( z_j' = \lambda (z_j - x_j) \). Hence we see that \( 0 \leq p \cdot z_j' = p \cdot \lambda (z_j - x_j) \) or \( p \cdot x_i \leq p \cdot z_i \).

Similarly, we can show that \( p \cdot x_i = p \cdot e_i \) for all \( i \in I \).

Suppose that \( G(x; y) = \mathbb{R}^l \). Then it is easy to see that \( 0 \leq p \cdot z \) for all \( z \in G(x; y) \) implies \( p = 0 \). Hence, \( p \cdot x_i \leq p \cdot z_i \) for all \( z_i \in \text{cl} P_i(x) \cap \text{cl} V(x; y) \) and all \( i \in I \) and \( p \cdot x_i = p \cdot e_i \) for all \( i \in I \) implies \( p = 0 \), which is impossible.

\[ \square \]

This lemma shows that for points \( x \) and \( y \) in \( X \), \( G(x; y) \neq \mathbb{R}^l \) is necessary and sufficient for the existence of a nonzero vector \( p \in \mathbb{R}^l \) which supports \( \text{cl} P_i(x) \cap \text{cl} V(x; y) \) at \( x_i \) and moreover, satisfies \( x_i \in \text{cl} B_i(p) \) for all \( i \in I \). This result leads to the following consequence.

**Proposition 3.2.1:** For any \( x \in X \), \( x \in F_1 \) and \( G(x; x) \neq \mathbb{R}^l \) if and only if there exists a nonzero \( p \in \mathbb{R}^l \) such that \( (p, x) \) is an equilibrium of the economy.

**Proof:** First we claim that \( \text{co}(\text{cl} P_i(x) - \{x_i\}) = \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \). To prove it, we need to show that \( P_i(x) - \{x_i\} \subset \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \). Let \( z_i \in P_i(x) - \{x_i\} \). Since \( x_i \in \text{co}(\text{cl} P_i(x) \cap \text{cl} V(x; x)) \), there exists \( \alpha \in [0, 1] \) such that \( x_i + \alpha z_i \in \text{cl} V(x; x) \). Let \( z_i' \) be the point in \( P_i(x) \) such that \( z_i = z_i' - x_i \). Since \( x_i \in \text{cl} P_i(x) \) and \( P_i(x) \) is convex, we have \( x_i + \alpha z_i = \alpha z_i' + (1 - \alpha) x_i \in \text{cl} P_i(x) \) and therefore, \( x_i + \alpha z_i \in \text{cl} P_i(x) \cap \text{cl} V(x; x) \). It implies that \( z_i \in \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \) and therefore, \( P_i(x) - \{x_i\} \subset \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \). On the other hand, \( \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \subset \text{co}(\text{cl} P_i(x) - \{x_i\}) \). Since \( P_i(x) - \{x_i\} \subset \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \), we must have \( \text{co}(\text{cl} P_i(x) - \{x_i\}) \subset \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \). Therefore we conclude that \( \text{co}(\text{cl} P_i(x) - \{x_i\}) = \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \).

By Lemma 3.2.1, we see that \( p \cdot x_i \leq p \cdot z_i' \) for all \( z_i' \in \text{cl} P_i(x) \cap \text{cl} V(x; x) \) and for all \( i \in I \), and \( p \cdot x_i = p \cdot e_i \) for all \( i \in I \). Recalling that \( \text{co}(\text{cl} P_i(x) - \{x_i\}) = \text{co}((\text{cl} P_i(x) \cap \text{cl} V(x; x)) - \{x_i\}) \).
Suppose that \((p,x)\) is an equilibrium of the economy. Then \(x \in F_1\). We have only to show that \(G(x; x) \neq \mathbb{R}^l\). Recall that

\[
G(x; x) = \sum_{i \in I} \text{co}(cP_i(x) \cap cV(x; x)) - \{x_i\} + \sum_{i \in I} L_i(x_i).
\]

Since \((p,x)\) is an equilibrium of the economy, it is easy to check that \(p \cdot x_i \leq p \cdot z_i\) for all \(z_i \in cP_i(x)\) and all \(i \in I\), and \(p \cdot x_i = p \cdot e_i\) for all \(i \in I\). Recalling that \(\text{co}(cP_i(x) - \{x_i\}) = \text{co}(cP_i(x) \cap cV(x; x)) - \{x_i\})\), we see that \(p \cdot x_i \leq p \cdot z_i\) for all \(z_i \in cP_i(x) \cap cV(x; x)\) and all \(i \in I\). By Lemma 3.2.1, we conclude that \(G(x; x) \neq \mathbb{R}^l\). \(\square\)

Proposition 3.2.1 shows that for a feasible allocation \(x \in F_1\) which satisfies \(G(x; x) \neq \mathbb{R}^l\), the set \(G(x; x)\) characterizes the conditions which are fulfilled in equilibrium.

This observation can give an idea about how to deal with the existence problem in the case where \(F\) may not be bounded. When the consumption sets are not bounded, traditional approaches to the existence proof rely on the truncation method introduced by Debreu (1959). Moreover, truncations of the consumption sets need to be taken sequentially so that the whole consumption sets can be covered in the limit when \(F_1\) is not bounded. The idea is to take advantage of information elicited from each truncated economy. By Theorem 2.4.1, each economy with the truncated consumption sets has an equilibrium. Let \(\{(p^n, z^n)\}\) in \(C_1 \times X\) be a sequence of equilibria for the truncated economies. If \(\{z^n\}\) is bounded, then by the same argument used in the proof of Theorem 2.4.1 we can show that its limit point constitutes an equilibrium. Otherwise, \(\{z^n\}\) fails to give information about equilibrium allocations. In this case, however, \(\{p^n\}\) turns out to convey useful information about the supporting property of equilibrium prices. For example, consider the case that for some \(y' \in F_1\), \(P_i(y') \subset G(z^n; y')\) for all \(n\) and all \(i \in I\). Then by the same argument used in the proof of Lemma 3.2.1, we can show that each \(p^n\) supports \(P_i(y')\) at \(z^n_i\). Moreover, by Proposition 3.2.1 the pair \((p, y')\) is an equilibrium of the economy \(E\) where \(p\) is the limit point of \(\{p^n\}\). These observations lead to the following condition which turns out to be a generalization of the no arbitrage conditions.
B6. For any sequence \( \{x^n\} \) in \( \tilde{F} \), there exist a subsequence \( \{x^{nk}\} \), a sequence \( \{y^{nk}\} \) and a point \( y \in F \) such that \( y^{nk} \to y \) and \( P_i(y^{nk}) \subset G(x^{nk};y) + \{x_i^{nk}\} \) for all \( i \in I \) and for all \( n_k \).

The reason why we introduce the sequence \( \{x^n\} \) in B6 is we need to take advantage of information elicited from a sequence of truncated economies. Suppose that B6 holds. If \( \{x^n\} \) is a sequence of equilibria for the truncated economies, then by Lemma 3.2.1 there exists a sequence of nonzero prices \( \{p^{nk}\} \) in \( C_1 \) such that \( p^{nk} \) supports \( P_i(y^{nk}) \) at \( x^{nk} \). As shown later, the limit point \((p, y)\) of \( \{(p^n, y^{nk})\} \) become an equilibrium of the economy. If \( x^n \) does not satisfies the equilibrium conditions of a truncated economy for some \( n \), then we may have \( G(x^n; y) = \mathbb{R}^l \). In this case, the relation \( P_i(y^{nk}) \subset G(x^{nk};y) + \{x_i^{nk}\} \) trivially holds for all \( i \in I \) but the existence of nonzero prices which support \( P_i(y^{nk}) \) for all \( i \in I \) is not warranted.

We show that B6 subsumes the well-known conditions of the literature as a special case.

**Lemma 3.2.2:** Suppose that B1-B5 holds and \( x_i \in clP_i(x) \) for all \( x \in X \) and for all \( i \in I \).\(^{14}\) Then each of the following conditions implies B6.

(i) \( F \) is bounded.

(ii) the CPP condition holds.

**Proof:** (i) Let \( \{x^n\} \) be a sequence in \( \tilde{F} \). Since \( \tilde{F} \) is bounded, it has a subsequence \( \{x^{nk}\} \) convergent to some point \( x \in F \). For all \( n_k \), let \( y^{nk} = x^{nk} \) and \( y = x \). Since \( x_i^{nk} \to y \), without loss of generality we can assume that \( x_i^{nk} \in V_i(y) \) for all \( i \in I \) and for all \( n_k \). Thus \( x_i^{nk} \in V(x^{nk};y) \).

We claim that \( P_i(x^{nk}) - \{x_i^{nk}\} \subset co((clP_i(x^{nk}) \cap clV(x^{nk};y)) - \{x_i^{nk}\}) \). Let \( z_i \) be a point in \( P_i(x^{nk}) - \{x_i^{nk}\} \). Then there exists \( z_i' \in P_i(x^{nk}) \) such that \( z_i' = z_i + x_i^{nk} \). Since \( x_i^{nk} \in clP_i(x^{nk}) \), there exists an arbitrarily small vector \( \epsilon_i \in \mathbb{R}^l \) such that \( x_i^{nk} + \epsilon_i \in P_i(x^{nk}) \). By convexity, \( \alpha(z_i + x_i^{nk}) + (1 - \alpha)(x_i^{nk} + \epsilon_i) \in P_i(x^{nk}) \) or \( \alpha z_i + (1 - \alpha) \epsilon_i + x_i^{nk} \in P_i(x^{nk}) \) for all \( \alpha \in [0,1] \). Since \( x_i^{nk} \in V(x^{nk};y) \) and \( V(x^{nk};y) \) is open, there exists \( \alpha' \in (0,1] \) such that \( \alpha' z_i + (1 - \alpha') \epsilon_i + x_i^{nk} \in V(x^{nk};y) \). By letting \( \epsilon_i \to 0 \), we see that \( \alpha' z_i + x_i^{nk} \in clP_i(x^{nk}) \) and \( \alpha' z_i + x_i^{nk} \in clV(x^{nk};y) \). Thus \( \alpha' z_i \) is in \( (clP_i(x^{nk}) \cap clV(x^{nk};y)) - \{x_i^{nk}\} \) and therefore, \( z_i \) is in \( co((clP_i(x^{nk}) \cap clV(x^{nk};y)) - \{x_i^{nk}\}) \).

Recalling that \( 0 \in (clP_i(x^{nk}) \cap clV(x^{nk};y)) - \{x_i^{nk}\} \) and \( y_i^{nk} = x_i^{nk} \) for all \( i \in I \), we see

\(^{14}\) As mentioned earlier in this section, the latter condition does no harm to the existence result as far as B4 is satisfied.
that \( P_i(y^{n_k}) - \{x_i^{n_k}\} \subset co((cl P_i(x^{n_k}) \cap cl V(x^{n_k}; y)) - \{x_i^{n_k}\}) \subset G(x^{n_k}; y) \).

(ii) Suppose the CPP condition holds. Let \( \{x^n\} \) be a sequence in \( \bar{F} \). Then there exist a subsequence \( \{x^{n_k}\} \), a sequence \( \{y^{n_k}\} \) in \( X \) and a point \( y \in F \) such that \( y^{n_k} \to y \) and \( y_i^{n_k} \in Q_i(x_i^{n_k}) \) for all \( i \in I \) and \( n_k \).

By transitivity, \( Q_i(y_i^{n_k}) \) is a subset of \( Q_i(x_i^{n_k}) \). By the same argument as used in (i), we see that for all \( n_k \) and all \( i \in I \),

\[
Q_i(y^{n_k}) - \{x_i^{n_k}\} \subset Q_i(x^{n_k}) - \{x_i^{n_k}\} \\
\subset co((cl Q_i(x^{n_k}) \cap cl V(x^{n_k}; y)) - \{x_i^{n_k}\}) \\
\subset G(x^{n_k}; y).
\]

\( \square \)

When preferences are numerically representable, Allouch (2002) shows that the CPP condition is equivalent to the compactness of the utility set which is generated by \( \bar{F} \). Since the no arbitrage conditions used by Werner (1987), Page (1987) or Dana et al. (1999) implies the compactness of the utility set, by Lemma 3.2.2 they also implies B6. But the converse is not true as illustrated below.

As illustrated in Example 3.1.1, the notion of arbitrage may not be appropriate for studying equilibrium in economies where agents are vulnerable to the non-transitivity of preferences. We show that the economy of Example 3.1.1 satisfies B6. Observe that \( Q_i = \{(a, b) \in \mathbb{R}^2 : b \geq 0 \} \) and \( Q_2 = \{(a, b) \in \mathbb{R}^2 : a \geq 0 \} \). For some point \( (a, b) \in \mathbb{R}^2 \), set \( x_1 = (-a, b) \) and \( x_2 = (a, -b) \). Since \( x_1 + x_2 = 0 \), \( \{x_1, x_2\} \) is in \( F_1 \) and \( x_1 \) and \( x_2 \) are on the boundary of the ball \( V(x) \) as shown in Figure 3. It is easy to check that \( x \) is in \( F \) if and only if \( a \geq 0 \) and \( b \geq 0 \). From now on, we assume that \( (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\} \). Then \( x \) is in \( F \). Let \( y \) be a point in \( X \) and assume that \( V_1(y) \subset V(x) \).

\(<\text{Insert Figure 3}>\)

It is easy to see that \( \sum_{i=1}^2 co([cl Q_i(x_i) \cap cl V(x; y)) - \{x_i\}] = \mathbb{R}^2_+ \) and \( L_i(x_i) \) is the line through \( x_i \) and the origin for all \( i \in I \). Thus \( G(x; y) \) is the halfspace with the boundary \( L_1(x_1) \). Since \( e_i = (0, 0) \) and \( Q_i(e_i) = \mathbb{R}^2_+ \), it follows that \( Q_i(e_i) - \{e_i\} \subset G(x; y) \) for all \( i \).

For each \( n \), set \( x_1^n = (-a^n, b^n) \) and \( x_2^n = (a^n, -b^n) \) for some \( a^n \) and \( b^n \) in \( \mathbb{R} \). We assume that \( x^n = (x_1^n, x_2^n) \) is in \( \bar{F} \) for all \( n \). Then we have \( a^n \geq 0 \) and \( b^n \geq 0 \). Let \( \{x^{n_k}\} \) be a subsequence of \( \{x^n\} \). If \( \{(x_{n_k}^1, x_{n_k}^2)\} \) is bounded, then by the same argument used in
proof of Lemma 3.2.2 we see that B6 is fulfilled. Thus, we have only to check the case where
\(a^n_k + b^n_k \to \infty\). Set \(y^n_k = e_i\) for all \(n_k\) and \(y_i = e_i\) for all \(i = 1, 2\), then \(V_1(y) \subset V(x^n_k; y)\). Thus, by applying the aforementioned results we see that \(Q_i(e_i) - \{e_i\} \subset G(x^n_k; y)\) for all \(n_k\) and all \(i \in I\). Thus B6 is fulfilled in this example.

3.3 Main Existence Theorems

With all these preliminary results out of the way, we can now turn to the main existence theorems of this paper. They are a generalization of all the existence results in the literature in several respects. In particular, preferences need not be transitive and moreover, they are interdependent and satisfy the condition B3 of weak continuity on preferences. More importantly, B6 is much weaker than the no arbitrage conditions used in the literature. We will also demonstrate that a weaker form of B6 is necessary for the existence of equilibrium whether preferences are transitive or not.

**Theorem 3.3.1:** Suppose that \(E\) satisfies the assumptions B1-B5. Then there exists an equilibrium \((p, x) \in C_1 \times X\) of the economy \(E\) if \(E\) satisfies B6.

**Proof:** As mentioned earlier in this section, without loss of generality we can assume that \(E\) satisfies B5-1 instead of B5. (For details, we refer the reader to the proof of Theorem 2.4.1.) Then \(P_i\) is convex-valued, and for all \(x \in X, x_i \in clP_i(x)\).

Let \(\{K^n\}\) denote a sequence of increasing closed balls centered at the origin in \(\mathbb{R}^l\) such that \(\mathbb{R}^l \subset \bigcup_{n=1}^{\infty} K^n\). We can take \(K^1\) to be a sufficiently large ball that \(e_i\) is contained in the interior of \(K^1\) for all \(i \in I\). Since \(K^n\) is increasing \(e_i\)'s are contained in the interior of \(K^n\) for all \(n\).

For each \(n\) and \(i \in I\), we define the sets
\[
X^n_i = X_i \cap K^n, \\
X^n = \prod_{i \in I} X^n_i,
\]
and for all \(x \in X^n\) and for all \(p \in C\),
\[
P^n_i(x) = P_i(x) \cap K^n, \\
B^n_i(p) = \{x_i \in X^n_i : p \cdot x_i < p \cdot e_i\}.
\]
For each \(n\), let \(E^n = \{(X^n_i, e_i, p^n_i) : i \in I\}\) denote the truncated economy of \(E\).
Thus by the existence result of Theorem 2.4.1, there exists an equilibrium $(p^n, x^n) \in C_1 \times X^n$ of the economy $E^n$ for each $n$, i.e., $(p^n, x^n)$ satisfies the following conditions; $x^n \in \hat{F}$, $\|p^n\| = 1$, and for all $i \in I$,

(a) $p^n \cdot x^n_i = p^n \cdot e_i$ and
(b) $P^n_i(x^n) \cap cl B^n_i(p^n) = \emptyset$.

In particular, (a) implies that $p^n \in H(x^n)$ for all $n$. Since $\|p^n\| = 1$ for all $n$, without loss of generality we may assume that $p^n$ converges to some point $p \in C_1$.

Recalling that $V(x^n)$ is the open ball with radius which is equal to the maximum of $\|x^n_i\|$’s over $i \in I, V(x^n)$ is a subset of $K^n$ for all $n$. If $clV(x^n)$ is in the interior of $K^n$ for some $n$, then $x^n_i$ is in the interior of $K^n$ for all $i \in I$. By the same argument of Step 5 of the proof of Theorem 2.4.1, $(p^n, x^n)$ is an equilibrium for $E$ and in this case, we are done. Thus, we only need to examine the case that for each $n$ there exists some $i_n \in I$ such that $x^n_{i_n}$ is in the boundary of $K^n$ or $clV(x^n_{i_n}) = K^n$. This has two implications. First, $\|x^n\|$ increases to infinity and therefore, $\{x^n\}$ has no bounded subsequences. Second, $P^n_i(x^n) = P_i(x^n) \cap cl V(x^n)$.

We claim that

$$cl P_i(x^n) \cap cl V(x^n) = \begin{cases} cl P^n_i(x^n), & \text{if } P^n_i(x^n) \neq \emptyset \\ \{x^n_i\}, & \text{if } P^n_i(x^n) = \emptyset. \end{cases}$$

Suppose that $P^n_i(x^n) \neq \emptyset$. Then $P^n_i(x^n)$ has the nonempty interior in $\mathbb{R}^l$ because $P_i(x^n)$ is open. Clearly, $int P^n_i(x^n) \subset P_i(x^n)$ and $int P^n_i(x^n) \subset V(x^n)$ and therefore, $int P^n_i(x^n) \subset P_i(x^n) \cap V(x^n)$. It implies that $cl P^n_i(x^n) \subset cl (P_i(x^n) \cap V(x^n)) \subset cl P_i(x^n)$. Since $P^n_i(x^n) \subset cl P_i(x^n) \cap cl V(x^n)$, trivially we have $cl P^n_i(x^n) \subset cl P_i(x^n) \cap cl V(x^n)$. Therefore we have $cl P_i(x^n) \cap cl V(x^n) = cl P^n_i(x^n)$.

Suppose that $P^n_i(x^n) = P_i(x^n) \cap cl V(x^n) = \emptyset$. Then $cl P_i(x^n) \cap cl V(x^n)$ is a close, convex set with the empty interior in $\mathbb{R}^l$. We claim that $cl P_i(x^n) \cap cl V(x^n) = \{x^n_i\}$. Since $x^n_i$ is in the boundary of $P_i(x^n)$ and of $V(x^n)$, clearly it is in $cl P_i(x^n) \cap cl V(x^n)$.

Suppose that there is a point $z_i$ in $cl P_i(x^n) \cap cl V(x^n)$ with $z_i \neq x^n_i$. Let $\alpha$ be a number in $(0, 1)$ and we set $z_i(\alpha) = \alpha x^n_i + (1 - \alpha)z_i$. Since $cl P_i(x^n) \cap cl V(x^n)$ is convex, it contains $z_i(\alpha)$. Recalling that $x^n_i$ and $z_i$ is in $cl V(x^n)$ and $cl V(x^n)$ is strictly convex, $z_i(\alpha)$ is in the open set $V(x^n)$. On the other hand, $z_i(\alpha) \in cl P_i(x^n)$. Thus $z_i(\alpha)$ is in $cl P_i(x^n) \cap V(x^n)$. Since $V(x^n)$ is open, it implies that $cl P_i(x^n) \cap cl V(x^n)$ has the nonempty interior, which is impossible.
**Step 2:** By B6 there exist a subsequence \( \{x^{nk}\} \), a sequence \( \{y^{nk}\} \) and a point \( y \in F \) such that \( y^{nk} \to y \) and \( P_i(y^{nk}) \subseteq G(x^{nk}; y) + \{x^{nk}_i\} \) for all \( i \in I \) and for all \( n_k \). Since \( \{x^{nk}\} \) is unbounded, without loss of generality we can assume that \( V(x^{nk}) = V(x^{nk}; y) \) and therefore, \( P^n_i(x^{nk}) = P_i(x^{nk}) \cap cI \{V(x^{nk}; y)\} \) for all \( n_k \) and all \( i \in I \). It follows from Step 1 that

\[
clP_i(x^{nk}) \cap cl I\{V(x^{nk}; y)\} = \begin{cases} 
    clP^n_i(x^{nk}), & \text{if } P^n_i(x^{nk}) \neq \emptyset \\
    \{x^{nk}_i\}, & \text{if } P^n_i(x^{nk}) = \emptyset.
\end{cases}
\]

We claim that for all \( z \in G(x^{nk}; y) \), \( p^{nk} \cdot z \geq 0 \). For each \( n_k \), let \( J^{nk} \) denote the set \( \{i \in I : P^n_i(x^{nk}) \neq \emptyset\} \). Then we see that for all \( n_k \),

\[
G(x^{nk}; y) = \sum_{i \in I} cO((clP_i(x^{nk}) \cap cl I\{V(x^{nk}; y)\}) - \{x^{nk}_i\}) + \sum_{i \in I} L_i(x^{nk}_i) = \sum_{i \in J^{nk}} cO(clP^n_i(x^{nk}) - \{x^{nk}_i\}) + \sum_{i \in I} L_i(x^{nk}_i).
\]

Let \( z \) be a point in \( G(x^{nk}; y) \). Then there exist vectors \( z_i \in clP^n_i(x^{nk}) \) and numbers \( \lambda_i \geq 0 \) for all \( i \in J^{nk} \), and numbers \( \mu_i \) for all \( i \in I \) such that \( z = \sum_{i \in J^{nk}} \lambda_i (z_i - x^{nk}_i) + \sum_{i \in I} \mu_i (x^{nk}_i - e_i) \). Since \( z_i \in clP^n_i(x^{nk}) \), there exists \( \epsilon^{nk}_i \) such that \( z_i + \epsilon^{nk}_i \in P^n_i(x^{nk}) \). By the equilibrium condition (b) in Step 1, we see that \( p^{nk} \cdot x^{nk}_i = p^{nk} \cdot (z_i + \epsilon^{nk}_i) \). Passing to the limit we have \( p^{nk} \cdot x^{nk}_i \leq p^{nk} \cdot z_i \). By the equilibrium condition (b) in Step 1, we know that \( p^{nk} \cdot x^{nk}_i = p^{nk} \cdot e_i \) for all \( n_k \) and all \( i \in I \). These results lead to the following result.

\[
p^{nk} \cdot z = \sum_{i \in J^{nk}} \lambda_i (p^{nk} \cdot (z_i - x^{nk}_i)) + \sum_{i \in I} \mu_i (p^{nk} \cdot (x^{nk}_i - e_i)) \geq 0.
\]

Let \( v_i \) be a point in \( P_i(y^{nk}) \). Since B6 implies \( P_i(y^{nk}) - \{x^{nk}_i\} \subseteq G(x^{nk}; y) \), it follows from the previous results that for all \( i \in I \) and all \( n_k \),

\[
p^{nk} \cdot x^{nk}_i \leq p^{nk} \cdot v_i.
\]

Recalling that \( y^{nk}_i \in \partial P_i(y^{nk}) \), we must have \( p^{nk} \cdot x^{nk}_i \leq p^{nk} \cdot y^{nk}_i \). Since \( p^{nk} \cdot x^{nk}_i = p^{nk} \cdot e_i \), it implies that for all \( i \in I \), \( p^{nk} \cdot e_i \leq p^{nk} \cdot y^{nk}_i \). Passing to the limit, we obtain \( p \cdot e_i \leq p \cdot y_i \). Since \( y \in F \), we see that \( p \cdot \sum_{i \in I} y_i = p \cdot \sum_{i \in I} e_i \) and therefore, for each \( i \in I \), \( p \cdot y_i = p \cdot e_i \).

**Step 3:** To complete the proof we must show that \( P_i(y) \cap cl B_i(p) = \emptyset \) for all \( i \). Suppose otherwise. Then there exists \( z_i \in P_i(y) \cap cl B_i(p) \) for some \( i \). Since \( P_i \) is open-valued, we have \( z_i \in P_i(y) \cap B_i(p) \) implying that \( p \cdot z_i < p \cdot e_i \). On the other hand, by Lemma 4.2 of Yannelis (1987) the correspondence \( P_i \cap B_i \) defined by \( P_i(x) \cap B_i(p) \) for all \( (p, x) \in C \times X \) is
lower semi-continuous. Therefore there exists \(z_i^{n_k} \in P_i(y^{n_k}) \cap B_i(p^{n_k})\) for each \(n_k\) which converges to \(z_i\). In particular, it implies that
\[
p^{n_k} \cdot z_i^{n_k} < p^{n_k} \cdot e_i.
\]
Since \(z_i^{n_k} \in P_i(y^{n_k})\), by Step 2 we must see that
\[
p^{n_k} \cdot e_i = p^{n_k} \cdot x_i^{n_k} \leq p^{n_k} \cdot z_i^{n_k},
\]
which leads to a contradiction.

**Step 4:** By the results of Step 1-Step 3, we see that \(y \in F_1, \|p\| = 1\), and \(y_i \in clB_i(p)\) and \(P_i(y) \cap clB_i(p) = \emptyset\) for all \(i\). Therefore we conclude that \((p, y)\) is an equilibrium of \(E\).

Now we turn to a necessary and sufficient condition for the existence of equilibrium.

**B6’.** There exists some sequence \(\{x^n\}\) in \(F\) which admits a subsequence \(\{x^{n_k}\}\), a sequence \(\{y^{n_k}\}\) and a point \(y \in F\) such that (i) \(y^{n_k} \to y\), (ii) \(G(x^{n_k}; y) \neq \mathbb{R}^l\), and (iii) \(P_i(y^{n_k}) \subset G(x^{n_k}; y) + \{x_i^{n_k}\}\) for all \(i \in I\) and for all \(n_k\).

This condition is a weaker version of \(B6\). Specifically, ‘any sequence’ is replaced by ‘some sequence’ which satisfies the condition (ii) of B6’.\(^{16}\)

**Theorem 3.3.2:** Suppose that \(E\) satisfies the assumptions B1-B5. Then it has an equilibrium if and only if B6’ holds true.

**Proof:** (\(\Rightarrow\)) Suppose that there exist a sequence \(\{x^n\}\) in \(F\) which admits a subsequence \(\{x^{n_k}\}\), a sequence \(\{y^{n_k}\}\) and a point \(y \in F\) such that \(y^{n_k} \to y\), \(G(x^{n_k}; y) \neq \mathbb{R}^l\) and \(P_i(y^{n_k}) \subset G(x^{n_k}; y) + \{x_i^{n_k}\}\) for all \(i \in I\) and for all \(n_k\). Since \(G(x^{n_k}; y) \neq \mathbb{R}^l\), by Lemma 3.2.1 there exists a nonzero point \(p^{n_k}\) such that for all \(z \in G(x^{n_k}; y)\), \(p^{n_k} \cdot z \geq 0\), i.e., \(p^{n_k} \cdot x_i^{n_k} \leq p^{n_k} \cdot z_i\) for all \(z_i \in clP_i(x^{n_k}) \cap clV(x^{n_k}; y)\) and all \(i \in I\), and \(p^{n_k} \cdot x_i^{n_k} = p^{n_k} \cdot e_i\) for all \(i \in I\). Without loss of generality we assume that \(\|p^{n_k}\| = 1\) for all \(n_k\). Then it has a subsequence convergent to some point \(p \in C_1\).

Since \(P_i(y^{n_k}) - \{x_i^{n_k}\} \subset G(x^{n_k}; y)\), \(y_i' \in P_i(y^{n_k})\) implies \(p^{n_k} \cdot x_i^{n_k} \leq p^{n_k} \cdot y_i'\). Recalling that \(y_i^{n_k}\) is in the boundary of \(P_i(y^{n_k})\), we have \(p' \cdot x_i^{n_k} \leq p' \cdot y_i^{n_k}\) for all \(n_k\) and all \(i \in I\).

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\(^{15}\)For details, see footnote 4.

\(^{16}\)In fact, B6’ is weaker than B6 because by Step 1 and 2 of the proof of Theorem 3.3.1, B6 implies the existence of a sequence \(\{x^n\}\) in \(F\) which satisfies (ii) of B6’.
Since $p^{n_k} \cdot x_{i}^{n_k} \leq p^{n_k} \cdot e_i$ and $x^{n_k} \in F_1$, by the same argument of Step 2 of the proof of Theorem 3.3.1 we obtain $p \cdot y_i = p \cdot e_i$ for all $i \in I$.

Again by the same argument of Step 3 of the proof of Theorem 3.3.1, we can verify that $P_i(y) \cap cl B_i(p) = \emptyset$ for all $i$. Therefore we conclude that $(p, y)$ is an equilibrium of the economy $E$.

$(\Rightarrow)$ Suppose that $E$ has an equilibrium $(p, x)$. For all $n$, we set $x^n = x$. Trivially $\{x^n\}$ converges to $x$. Let $\{x^{n_k}\}$ be a subsequence and set $y^{n_k} = x^{n_k}$ for all $n_k$. Then $\{y^{n_k}\}$ converges to $x$. We must notice that $x = x^{n_k} = y^{n_k}$ for all $n_k$.

By the same argument used in proving Proposition 3.2.1, we obtain $P_i(x^{n_k}) - \{x_i^{n_k}\} \subset co((cl P_i(x^{n_k}) \cap cl V(x^{n_k}; y)) - \{x_i^{n_k}\})$ and therefore, $P_i(x^{n_k}) \subset G(x^{n_k}; y) + \{x_i^{n_k}\}$ for all $i \in I$ and for all $n_k$. Since $(p, x^{n_k})$ is an equilibrium, by Proposition 3.2.1 we have $G(x^{n_k}; y) \neq \mathbb{R}^l$ for all $n_k$ and therefore, B6' is satisfied. □
REFERENCES


