Cones and conditions for existence of equilibrium in economies with short selling.

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Abstract

In this note we study four different cone concepts used in the recent literature. In the case of no half lines in indifference surfaces (NHL) we show that the arbitrage cone, the recession cone of the preferred set, coincides with the Page-Wooders increasing cone and with the closure of Chichilnisky’s original global cone (cf., Chichilnisky 1995). The fact that a closure is required for this equivalence underlies the difficulties with Chichilnisky’s purported existence of equilibrium results using the global cone. From the equivalence of the increasing cone and the arbitrage cone in the NHL case, it follows that the existence of equilibrium result claimed in Chichilnisky (1997) is a special case of a more general result due to Werner (1987).

Under some special assumptions, in the half lines (HL) case, where the normalized set of gradients to each indifference surface is a closed set, we demonstrate that the global cone is determined by the union of the directions of the half lines in indifference surfaces. Under the same special assumptions for the HL case, the increasing cone, the global cone and the interior of the arbitrage cone all coincide. We identify a gap in Chichilnisky’s purported proof of existence of equilibrium in the HL case and discuss additional conditions that will enable a proof of her conjecture for this case.

1 Introduction

In this paper we study four cone concepts used in papers investigating existence of economic equilibrium in models with consumption sets unbounded below. The cones are the well known arbitrage cone (the recession cone of the preferred set), introduced into economic theory in Debreu (1959), the strict increasing cone, introduced in Page (1982) and Page and Wooders (1994), the increasing cone, introduced in Page and Wooders (1996), and the global cone, introduced in a number of papers due to G. Chichilnisky (C), including C (1995). Based on our findings concerning the properties
of cones, we return to the claims concerning existence of equilibrium in C (1997). We conclude that in the case of no half lines in indifference surfaces (NHL), in view of C’s special assumptions on the economy, the claimed existence of equilibrium in C (1997) is a special case of the existence result in Werner (1987). In the other case treated in C (1997), where the set of normalized gradients to indifference surfaces are required to be closed (the closed gradient or HL case), there is a severe gap in C’s purported proof of existence of equilibrium. The gap arises from a failure to recognize that, unless a function satisfies some continuity condition, even if the domain of a function is compact the function may not achieve a maximum or have a finite supremum. In the special case of two commodities we provide a proof of Chichilnisky’s conjecture for the HL case and, with an additional condition that permits proof of existence of equilibrium in the HL case.

Since providing necessary and sufficient conditions for existence of equilibrium in the HL case seems to be one of the main motivators for C’s numerous papers introducing conditions limiting arbitrage opportunities, we note that the assumptions required in C’s papers are already quite special, even without our addition condition. First, the HL (closed gradient) condition roughly requires that every indifference surface eventually becomes ‘flat’ (i.e., becomes a plane or half-line). In two-dimensional Euclidean space, this means that each indifference surface must contain two half-lines, one on its right hand side and the other on its left hand side. Finally, C also requires a condition she calls ‘uniform nonsatiation’. In the HL case, this condition dictates that the half-lines in indifference surfaces are parallel – they can neither ‘fan’ nor ‘squeeze’. Yet these conditions are apparently not enough; at present there is still no proof showing that C’s ‘limited arbitrage’ condition implies existence of equilibrium in the HL case. Our additional condition is that the arbitrage cone of each agent, the recession cone of the commodities preferred set to the endowment, is polyhedral.

Placing our research in the broader context of the literature on arbitrage in general equilibrium models, let us first discuss the role of the arbitrage cone. In asset market models this cone concept has played an important role in existence of equilibrium with consumption sets unbounded below (see, for example, Hart 1974, Milne 1981, Hammond 1983, and Page 1982,1987). More recently, in the context of general equilibrium models, the arbitrage cone has been used for the same purpose (see, for example, Werner 1987, Nielsen 1989, Page and Wooders 1996a,1996b and Dana, Magnien, and Le Van 1996).

As the literature referenced above concludes, it is not the possibility of unbounded trades but rather utility-increasing unbounded trades that presents difficulties for the existence of equilibrium. For example, Werner’s (1987) condition ensuring existence of equilibrium can be satisfied even when there exists mutually compatible, arbitrarily large trades, as long as for each agent these trades are in the lineality space corresponding to the set of commodity bundles preferred to agents’ endowment - and thus as long as these trades are utility non-increasing. Page (1982) based necessary

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1 An asset market model is a general equilibrium model with preferences determined by agents’ probability beliefs over asset returns and risk-aversion characteristics.

2 This was especially stressed in Page (1989).
and sufficient conditions for existence of equilibrium on the strict increasing cone—beginning at the endowment, those directions in which utility is strictly increasing. The insights of this literature, and in particular, of Werner (1987), perhaps motivated the definition of global cone in C (1995). The global cone is defined as the set of rays emanating from the agent’s endowment along which, not only does utility increase forever, but increases beyond the utility level of any other vector in the agent’s consumption set. The increasing cone, introduced in Page and Wooders (1996) is the set of directions emanating from the endowment in which utility eventually strictly increases.

Since it imposes more restrictive requirements than either of the arbitrage cone or the increasing cone, it is immediate that the global cone is contained in the arbitrage cone and also in the increasing cone. What is less clear is whether the global cone is significantly different (say, for example, by more than a closure) from the arbitrage cone and the increasing cone. In this paper, we provide examples showing that the closure of the global cone may be a strict subset of the increasing cone. Thus, except in special circumstances (see Dana, Magnien, and Le Van 1996 for example), conditions based on the global cones may not adequately limit arbitrage opportunities. In fact, Monteiro, Page and Wooders (1997,1998) provide counterexamples to some of C’s claims on existence of equilibrium. Nevertheless, the differences between the global cone and the increasing cone shed light on the structure of arbitrage cones and the problem of existence of equilibrium.

We also consider the relationship between the arbitrage cone, the increasing cone and the global cone in the context of the special assumptions of C’s papers. These assumptions ensure that indifference curves to do not ‘fan’ nor ‘squeeze,’ called uniform nonsatiation, and that, if an indifference surface contains half lines, the mapping from the set of points in an indifference surface to the set of normalized gradients of those points is closed, called the closed gradient condition. One interesting implication of the closed gradient condition, which we will demonstrate, is that if an agent’s preferences satisfy this condition, then the arbitrage cone is determined by the directions of the half-lines in his indifference surfaces. In addition, under these conditions the increasing cone and the global cone are equal and equal to the interior of the recession cone.

We conclude with further motivation for this paper and additional relationships to the literature.

2 Some definitions.

Some definitions are required. Let $K$ be a closed convex subset of $\mathbb{R}^L$. We say that $y \in \mathbb{R}^L$ is a direction of recession of $K$ if for any $x \in K$, $x + \lambda y \in K$ for $\lambda \geq 0$. Denote by $R(K)$ the set of all recession directions of $K$. The set $R(K)$ is called the recession cone of $K$. Now define $L(K) := -R(K) \cap R(K)$. The set $L(K)$ is called the lineality space of $K$.

Let $S$ be any subset of $\mathbb{R}^L$. The positive dual cone corresponding to $S$, denoted
by \( \Lambda^+(K) \), is given by
\[
\Lambda^+(S) := \{ p \in \mathbb{R}^L : \langle y, p \rangle > 0 \text{ for all } y \in S \}.
\] (1)

3 The economic model

Let \( (X_j, \omega_j, u_j(\cdot))_{j=1}^n \) denote an exchange economy. Each agent \( j \) has consumption set \( X_j \subset \mathbb{R}^L \) and endowment \( \omega_j \). The set \( X_j \) is assumed to be closed, convex and nonempty. The \( j \)th agent’s preferences over \( X_j \) are specified via a utility function \( u_j(\cdot) : X_j \rightarrow \mathbb{R} \). It is assumed that for each \( j \in N := \{1, \ldots, n\} \), \( X_j \) is closed and convex and \( \omega_j \in \text{int } X_j \) where “\( \text{int} \)” denotes “interior”. The preferred set of the \( j \)th agent at \( x \in X_j \) is given by
\[
P_j(x) := \{ x_0 \in X_j : u_j(x_0) > u_j(x) \}
\] (2) and the weak preferred set at \( x \in X_j \) is given by
\[
\bar{P}_j(x) := \{ x_0 \in X_j : u_j(x_0) \geq u_j(x) \}.
\] (3)

The set of feasible and individually rational allocations is given by
\[
F := \{ (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \text{ and } u_j(x_j) \geq u_j(\omega_j) \text{ for all } j \}.
\] (4)

3.1 Arbitrage cones, increasing cones, and global cones

The \( j \)th agent’s arbitrage cone at endowments \( \omega_j \in X_j \) is the closed convex cone containing the origin given by
\[
R(\bar{P}_j(\omega_j)) := \{ y \in \mathbb{R}^L : \text{for } x \in \bar{P}_j(\omega_j) \text{ and } \lambda \geq 0, x + \lambda y \in \bar{P}_j(\omega_j) \}
\] (5)

The arbitrage cone \( R(\bar{P}_j(\omega_j)) \) is the recession cone corresponding to the \( j \)th agent’s weak preferred set \( \bar{P}_j(\omega_j) \) at the agent’s endowment \( \omega_j \); this, and the fact that an arbitrage cone is closed and convex, follows from results in Rockafellar (1970, Section 8).

The strict increasing cone at \( x \in X_j \) is defined by \(^3\)
\[
I^*_j(x) := \{ y \in R(X_j) : x + \lambda y \in P_j(x + \mu y) \text{ for all } \mu \geq 0 \text{ and } \lambda > \mu \},
\] (6)

where \( x \) is called the origin of the cone. The strict increasing cone is the set of rays emanating from \( x \in X_j \) along which utility increases forever. In Page and Wooders

\(^3\)See also Page (1982), where the increasing cone was introduced to show necessary and sufficient conditions for the existence of equilibrium in an asset market model.
(1996a,b), the definition of the increasing cone is modified to allow thick indifference surfaces and given by

\[ I_j(x) := \left\{ y \in R(\bar{P}_j(\omega_j)) : \text{for all } \mu \geq 0, \text{there exists } \lambda > \mu \text{ such that } x + \lambda y \in P_j(x + \mu y) \right\}. \]  

(7)

The point \( x \) is called the origin of the increasing cone.

The global cone at endowments \( \omega_j \in X_j \) is defined by

\[ A_j(\omega_j) := \{ y \in X_j : \forall x \in X_j, \exists \lambda > 0 \text{ such that } u_h(\omega_j + \lambda y) > u(x) \}. \]  

(8)

Thus, as stated in the introduction, the global cone, introduced in C (1995,1997) and earlier papers, is the set of rays emanating from the endowment along which not only does utility increase forever, but it increases beyond the utility level of any other vector in the consumption space. In C (1997), however, the definition of the global cone is changed to

\[ G_j(\omega_j) := \{ y \in R(X_j) \not\in \max_{\lambda \geq 0} u_j(\omega_j + \lambda x) \}. \]  

(9)

C writes that “this cone is identical to the global cone \( G_h(\Omega_h) \) in C (1995a).” Later, we provide an example showing that, in general, the cones \( G \) and \( A \) are distinct. In fact, in economic models such as Werner (1987), Page and Wooders (1996a,b,2000) and Dana, Le Van and Magnien (1999), the closure of the global cone \( A_j \) may be a proper subset of the increasing cone \( I_j = G_j \).

In view of the following Proposition and to avoid having the same terminology for two distinct concepts, we will continue to call the cone \( A_j \) the global cone and the cone \( I_j \) the increasing cone. Our first proposition needs no proof.

Proposition 1:. The cone \( G_j(\omega_j) \), defined by (9), is equal to the increasing cone \( I_j(\omega_j) \) defined by (7).

### 3.2 Assumptions

We shall maintain the following assumptions throughout: for each agent \( j = 1, \ldots, n \),

(A.1) \( u_j(\cdot) \) is continuous, quasi-concave and nondecreasing.\(^4\)

(A.2) Global nonsatiation at rational allocations: For any rational allocation \((x_1, \ldots, x_n) \in F, P_j(x_j) \neq \emptyset.\) See the Proposition 3.1.

\(^4\)That the utility function \( u_j(\cdot) : X_j \rightarrow \mathbb{R} \) is quasi-concavity typically means (and means here) that if \( x, x^0 \) are in \( X_j \) and \( \lambda \in [0, 1] \), then \( u_j(\lambda x + (1 - \lambda)x^0) \geq \min\{u_j(x), u_j(x^0)\} \). Although Chichilnisky (1997) uses the term ‘quasi-concave’ no definitition is provided. This is not important, however, for the purposes of the current paper.
For some of our examples and results, we shall appeal to the following additional assumptions concerning an agent utility function. To spare the reader unhelpful notation, unless more than one agent is involved, we leave off the subscripts denoting the agent. Let $u(\cdot) : X \to \mathbb{R}$ be the agent’s utility function:

$u(\cdot)$ is monotonic:

For $x, x'$ in $X$ with $x < x'$ it holds that $u(x) \leq u(x')$ and $x << x'$ implies $u(x) < u(x')$.

(10)

$u(\cdot)$ is strictly monotonic:

For $x, x'$ in $X$ with $x < x'$ implies $u(x) < u(x')$.

(11)

$u(\cdot)$ is strictly quasi-concave:

For $x, x'$ in $X$ with $x \neq x'$ and $\lambda \in (0, 1)$, $u(\lambda x + (1-\lambda)x') > \min\{u(x), u(x')\}$.

(12)

$u(\cdot)$ satisfies weak strict quasi-concavity:

For $x, x'$ in $X$ with $u(x) \neq u(x')$ and $\lambda \in (0, 1)$, $u(\lambda x + (1-\lambda)x') > \min\{u(x), u(x')\}$.

(13)

$u(\cdot)$ satisfies uniformity of increasing cones:

For $x, x' \in X$, $I(x) = I(x') := I$.

(14)

$u(\cdot)$ is such that there exists useful trades:

For $x \in X$, $R(\hat{P}(x)) \setminus L((P(x)) \neq \emptyset$.

(15)

an assumption introduced by Werner (1987).

$u(\cdot)$ satisfies global uniformity of arbitrage cones, UAC: For $x, x' \in X$,

$R(\hat{P}(x)) = R(\hat{P}(x')) := R$.

(16)

$u(\cdot)$ satisfies no half lines, NHL:

There does not exist $x \in X$ and $y \in R(X)$, $y \neq 0$, such that $u(x) = u(x + \lambda y)$ for all $\lambda \geq 0$.

(17)

$u(\cdot)$ is $C^1$.

The utility function $u(\cdot)$ is continuously differentiable.

(18)

$u(\cdot)$ satisfies uniform nonsatiation, UNS: For $u(\cdot)$ continuously differentiable, there exists positive numbers $K$ and $\varepsilon$ such that $x \in X$

$\varepsilon < ||\nabla u(x)|| < K$.

(19)

\footnote{Note that for $x = (x_1, \ldots, x_L)$ and $y = (y_1, \ldots, y_L)$ in $\mathbb{R}^L$, $x < y$ if and only if $x_i \leq y_i$ holds for all $i$ and $x_i < y_i$ for at least one $i$.}
where \( \nabla \) denotes the gradient. This assumption was introduced into the literature on unbounded short sales by Chichilnisky (introduced in other contexts in prior literature).

**u(·) satisfies the closed gradients condition, CG:** For \( u(·) \) continuously differentiable, let \( \sigma_\alpha \) denote an indifference surface, i.e., \( \sigma_\alpha = \{x : u(x) = \alpha\} \), with \( \alpha \in \mathbb{R} \). Then

for any \( \alpha \in \mathbb{R} \), the map \( x \in \sigma_\alpha \rightarrow \nabla u(x)/\|\nabla u(x)\| \) is closed, \hspace{1cm} (20)

also introduced into the list of assumptions on economies with unbounded short sales by Chichilnisky.

\( u(·) \) is concave and

\[
\sup_{x \in X} u(x) = +\infty. \hspace{1cm} (21)
\]

The following remark is from Monteiro, Page and Wooders (2000). Informally, the remark implies that indifference surfaces neither squeeze together nor fan. See Figures 1 and 2 below.

**Remark 1.** For differentiable utility functions, uniform nonsatiation (19) implies

(a) For all \( r, s \in \mathbb{R} \) there exists \( N(r, s) \in \mathbb{R} \) such that \( tx \in u^{-1}(r) \)

\[
\Rightarrow \text{there exists } z \in u^{-1}(s) \text{ with } \|x - z\| \leq N(r, s);
\]

and

(b) \( N(\cdot, \cdot) \) is bounded on bounded sets. \hspace{1cm} (22b)

The proof is as follows:

The differential equation

\[
z'(w) = \nabla u(z(w)) \quad z(0) = x.
\]

has a solution for \( w \geq 0 \). Thus if \( r = u(x) \) and \( s = u(y) \) then since \( u(z(w)) = u(x) = \int_0^w \nabla u(z(w)) \cdot z'(w) dw = \int_0^w |\nabla u(z(w))|^2 dw \geq \varepsilon^2 w \) we have that if \( w \) satisfies \( u(z(w)) = s \), then defining \( y = z(w) \), it holds that \( s - r \geq \varepsilon^2 w \). From \( |y - x| = |z(w) - x| \leq |z'(w)|w \leq Kw \) we finally obtain \( s - r \geq \frac{\varepsilon^2 |y - x|}{K} \) and \( N(r, s) = \frac{|s - r|K}{\varepsilon^2} \).

**3.3 The assumptions of C (1997).**

In C (1997), to treat consumption sets unbounded below, the author assumes (A.1), that the preferences of an agent are representable by a continuous, quasi-concave and increasing function, uniformly nonsatiated (19), and satisfy either the condition that there are no half lines in indifference curves (17) or the closed gradient condition.
Note that uniform nonsatiation implies monotonicity. In addition, on page 464 of C (1997) the author appeals to Lemma 1 of her Economic Theory paper to proceed with her purported proof. The proof of the Lemma invoked, however, appeals to concavity of utility functions. In addition, C (1995, 1997) makes the fairly nonrestrictive assumption that \( \sup_{x \in X} u(x) = +\infty \) (21). Thus, besides standard assumptions of quasi-concavity and monotonicity, the major assumptions of C (1997) are:

- Concavity and \( \sup_{x \in X} u(x) = +\infty \) (21);
- Uniform nonsatiation (19);
- Either:
  - No half lines in indifference surfaces (17) or
  - The closed gradient condition (20)

The two assumptions, uniform nonsatiation and the closed gradient condition are quite powerful. In \( \mathbb{R}^2 \), they imply, for example, that if there are half-lines in an agent’s indifference surface, then there are half-lines in all the indifference surfaces of that agent and these are parallel; as depicted in Figure 1. Situations such as that of Figure 2 are ruled out.

The Figure below illustrates other sorts of preferences ruled out by uniform nonsatiation and the closed gradient condition.

We include here two lemmas concerning the global cone \( A \).

**Lemma 1.** (Dana, Le Van and Magnien 1999). Assume global nonsatiation. Then \( y \in A(\omega) \) if and only if there exists a sequence of positive real numbers \( \lambda_\nu \) such that

\[
\lim_{\lambda_\nu} u(\omega + \lambda_\nu y) \rightarrow \bar{u} := \sup_{x \in X} u(x).
\]

**Proof.** Immediate. \( \blacksquare \)
Lemma 2. (Dana, Le Van and Magnien 1999). Assume (A.1) and global nonsatiation. Then

\[ y \in A(\omega) \]

if and only if

\[ \{ y \in R(\bar{P}(\omega)), \lim_{\lambda \to \infty} u(\omega + \lambda y) = \overline{\pi} := \sup_{x \in X} u(x) \}. \]

Proof. \( \implies \): From Dana, Le Van and Magnien (1999) Lemma 4, for some \( \lambda_0 \geq 0 \), the function \( \phi(\lambda) = u(\omega + \lambda y) \) is nondecreasing on \([0, \lambda_0)\) and then increasing from \([\lambda_0, \infty)\). By the definition of the global cone \( A \), \( \phi(\lambda) \) must be nondecreasing from \([0, \infty)\). Thus, the limit of \( u(\omega + \lambda y) \) as \( \lambda \to \infty \) exists. From the proceeding Lemma, this limit equals \( \overline{\pi} \).

\[ \iff \text{Obvious.} \]

4 Relating the increasing cone, the global cone and the arbitrage cone.

4.1 Equivalence of the increasing cone and the strict increasing cone.

The following Proposition strengthens Proposition 1 of Monteiro, Page and Wooders (2000). We first require a Lemma.

Lemma 3. Let \( u(\cdot) : X \to \mathbb{R} \) be a continuous, quasi-concave utility function.

(i) Assume that for all \( x \in X \), \( P(x) = i^r(\bar{P}(x)) \). Then \( u(\cdot) \) satisfies weak strict quasi-concavity (13).

(ii) Assume that \( u(\cdot) \) satisfies uniform nonsatiation (19) and weak strict quasi-concavity (13). Then, for all \( x \in X \), the strictly preferred set equals the interior of the preferred set; that is, \( P(x) = i^r(\bar{P}(x)) \).

Proof: (i) Let \( x' \in P(x) \) and let \( \lambda \in (0, 1) \). Then \( x' \in i^r(\bar{P}(x)) \) by (A.1) and hence, \( \lambda x' + (1 - \lambda) x \in i^r(\bar{P}(x)) = P(x) \), that is, \( u(\lambda x' + (1 - \lambda) x) > u(x) \). Therefore \( u(\cdot) \) satisfies (13).

(ii) Let \( x' \in i^r(\bar{P}(x)) \) and assume that \( u(x) = u(x') \). By (19) there exists \( s \in X \) such that \( u(s) > u(x') \). From quasi-concavity of the utility function, it follows that \( s \in i^r(\bar{P}(x)) \). Thus, there exists a (relative) ball \( B(x') \) with centre \( x' \) satisfying \( B(x') \subset i^r(\bar{P}(x)) \). This implies that there exists \( x_1 \) between \( x' \) and \( s \) such that \( x_1 \in B(x') \). Select \( x_2 \) so that \( x' = \frac{1}{2}x_1 + \frac{1}{2}x_2 \), \( x_2 \in B(x') \). We then have \( u(x_1) > u(x') \) by weak strict quasi concavity. If \( u(x_2) = u(x) = u(x') \), then by (13), \( u(x') > u(x') \), which is impossible. If \( u(x_2) > u(x) \) then \( u(x') \geq \min(u(x_1), u(x_2)) > u(x) \), which is also impossible. Therefore \( u(x') > u(x) \), implying that \( x' \in P(x) \). Since \( P(x) \subset i^r(\bar{P}(x)) \), we have \( P(x) = i^r(\bar{P}(x)) \).
Proposition 2. Let \( u(\cdot) : X \to \mathbb{R} \) be a continuous, quasi-concave utility function. If \( u(\cdot) \) satisfies either:

1. uniform nonsatiation (19) and weak strict quasi-concavity (13),

or

2. monotonicity (10),

then

\[
I^*(\omega) = I(\omega) \text{ for all } x \in X.
\]

(Observe that monotonicity implies that \( X + R^+_4 \subset X \), which implies that \( i^*X \neq \emptyset \).)

Proof.

1. (a) \( I^*(\omega) \subset I(\omega) \). We prove the converse. Let \( y \in I(\omega) \) and let \( \mu_0 \) be given. Then there exists \( \mu_1 \) such that \( u(\omega + \mu_1 y) > u(\omega + \mu_0 y) \). By Lemma 3, \( \omega + \mu_1 y \in \bar{i}((\hat{P}(\omega + \mu_0 y)) \). Then for any \( \lambda \in [0, 1] \),

\[
(\lambda \mu_1 + (1 - \lambda)\mu_0)y + \omega \in \bar{i}'((\hat{P}(\omega + \mu_0 y)) = (\hat{P}(\omega + \mu_0 y)).
\]

Then \( u(\lambda \mu_1 + (1 - \lambda)\mu_0 y + \omega) > u(\omega + \mu_0 y) \). The result follows by induction.

(b) \( I(\omega) \subset I^*(\omega) \). Let \( y \in I(\omega) \) and let \( \mu_0 > 0 \) be given. Then, from the definition of \( I(\omega) \), there exists \( \mu_1, \bar{\mu} \) with \( \mu_1 > \bar{\mu} \geq \mu_0 \) such that \( u(\omega + \mu y) > u(\omega + \mu_0 y) \) for all \( \mu \in [\bar{\mu}, \mu_1] \). If \( \bar{\mu} > \mu_0 \), then \( \omega + \bar{\mu} y \in \bar{i}'(\hat{P}(\omega + \mu_0 y)) \). There exists a ball \( B(\omega + \bar{\mu} y) \) with centre \( \omega + \bar{\mu} y \) such that \( B(\omega + \bar{\mu} y) \subset \hat{P}(\omega + \mu_0 y) \). But one can choose \( \xi \in B(\omega + \bar{\mu} y) \) with \( \xi << \omega + \bar{\mu} y \). We obtain a contradiction: By monotonicity, \( u(\xi) < u(\omega + \bar{\mu} y) = u(\omega + \mu_0 y) \), while \( u(\xi) \geq u(\omega + \mu_0 y) \). Thus, \( \bar{\mu} = \mu_0 \), and \( u(\omega + \mu y) > u(\omega + \mu_0 y) \) for all \( \mu \in [\mu_1, \mu_0] \). 

Remark. Note that if \( u(\cdot) \) satisfies strict quasi-concavity (12), then \( u(\cdot) \) satisfies no half lines (17) automatically. Thus, it follows from Theorem 1 in Monteiro, Page, and Wooders (1997) that if \( u(\cdot) \) is strictly quasi-concave, then

\[
I(x) = R(\hat{P}(x)) \setminus \{0\}, \text{ for all } x \in X.
\]

4.2 Global uniformity of cones

As noted previously, global uniformity of arbitrage cones (16) is a consequence of the assumption of concavity of utility functions (see Rockafellar (1970), section 8). Global uniformity as a separate assumption was required in Werner (1987) to show that his condition limiting arbitrage is sufficient, as well as necessary, for existence of equilibrium. Page and Wooders (1996a,b) require the assumption of global uniformity of arbitrage cones to demonstrate that, with this assumption, their condition limiting arbitrage is necessary for existence of equilibrium.

Proposition 4. (Monteiro, Page and Wooders 2000). Let \( u(\cdot) : X \to \mathbb{R} \) be a continuous, quasi-concave utility function satisfying strict monotonicity (11). If \( u(\cdot) \)
satisfies global uniformity of arbitrage cones, (16), then \( \text{cl}I(x) = R(\hat{P}(x)) \) for all \( x \in X \) where “cl” denotes closure.

**Proof.** Suppose \( y \in R(\hat{P}(x)) \). Global uniformity implies that \( x' + ty \in \hat{P}(x') \) for every \( t > 0, x' \in X \). For any integer \( n \geq 1 \) define \( y^n = y + \frac{1}{n} (1, \ldots, 1) \). For any \( t > 0 \), \( u(x + (t+1)y^n) > u(x + ty^n + y) \geq u(x + ty^n) \). Thus \( y^n \in I(x) \). Hence \( y \in \text{cl}I(x) \). ■

**Remark.** Weak monotonicity is not sufficient in the Proposition above. Consider for example, \( u(\cdot): R^2 \to R \) given by \( u(x_1, x_2) = \max \{0, \min \{x_1, x_2\}\} \). This utility function is continuous, quasi-concave with increasing cone \( I(x_1, x_2) = R_2^+ \) for all \((x_1, x_2) \in R^2\), but \( R(\hat{P}(0)) = R^2 \).

If the agent’s utility function is concave, then global uniformity of arbitrage (16) holds automatically.

According to our next Proposition, if the agent’s utility function is only quasi-concave but satisfies uniform nonsatiation (19), then \( u(\cdot) \) will again satisfy global uniformity of arbitrage cones (16).

**Proposition 5.** (Monteiro, Page and Wooders 2000) Let \( u(\cdot): X \to R \) be a continuous, quasi-concave utility function. If \( u(\cdot) \) satisfies uniform nonsatiation (19) then \( u(\cdot) \) also satisfies global uniformity of arbitrage cones (16).

**Proof.** First, recall Remark 1. Now suppose \( y \in R(\hat{P}(\omega)) \). Then there exist a sequence \( \{y^n\}_n \subset \hat{P}(\omega) \) and \( \lambda^n \downarrow 0 \) such that \( \lambda^n x^n \to y \). If \( x \in X \) is given and if \( x^n \in \hat{P}(x) \) for infinitely many \( n \) then \( y \in R(\hat{P}(x)) \). If for some \( n^0, n \geq n^0 \) implies that \( x^n \notin \hat{P}(x) \) (i.e., \( u(x^n) < u(x) \)), define \( r = u(\omega) \) and \( s = u(x) \). By the first part of Remark 1, there is a sequence \( \{\tilde{x}^n\}_n \) with \( u(\tilde{x}^n) = u(x) \) such that \( |x^n - \tilde{x}^n| \leq N(u(x^n), s) \). By the second part of Remark 1, \( N(\cdot, \cdot) \) is bounded on bounded sets, so that \( \lambda^n N(u(x^n), s) \to 0 \). Thus \( |\lambda^n x^n - \lambda^n \tilde{x}^n| \to 0 \) implying that \( \lambda^n \tilde{x}^n \to y \). This in turn implies that \( y \in R(\hat{P}(x)) \). Since \( \omega \) and \( x \) are chosen arbitrarily, we conclude that the arbitrage cones are all equal. ■

**Remark.** If \( u(\cdot) \) is also satisfies strict monotonicity in addition to the conditions of Proposition 5, then by Proposition 4, we can conclude that under those conditions \( \text{cl}I(x) = R(\hat{P}(x)) \) for all \( x \in X \).

4.3 **Equivalence of the increasing cone and the recession cone with no half lines**

The following Proposition allows us to easily show that in the no half line case, the condition claimed in C (1997) is simply Werner’s (1987) condition. As we will illustrate by an example, the equality stated below between the increasing cone and the recession cone of the weakly preferred set cannot be extended to equality with the global cone \( A \). The example further explains the counterexample in Monteiro, Page and Wooders (1997) and why conditions for existence of equilibrium based on
C’s global cone cannot, except perhaps in special circumstances, be sufficient for existence of equilibrium.

**Proposition 6.** Let \( u(\cdot) : X \rightarrow \mathbb{R} \) be a continuous, quasi-concave utility function. If \( u(\cdot) \) satisfies uniform nonsatiation (19) and no half-lines, NHL (17), then \( u(\cdot) \) also satisfies

\[
I(\omega) = R(\hat{P}(x)) \setminus \{0\}.
\]

**Proof.**

We first prove that \( I(\omega) \subset R(\hat{P}(x)) \setminus \{0\} \). Let \( y \in I(\omega) \). It is obvious that \( y \neq 0 \). There are two exclusive cases. Case 1: For all \( \lambda \geq 0 \), \( u(\omega + \lambda y) \geq u(\omega) \). Then, by definition of \( R(\hat{P}(x)) \), \( y \in R(\hat{P}(x)) \). Case 2. There exists \( \lambda_0 \geq 0 \) such that \( u(\omega + \lambda_0 y) < u(\omega) \). Let \( f(\lambda) = u(\omega + \lambda y) \) for \( \lambda \geq 0 \). From Lemma 4 of Dana, Le Van and Magnien (1999), page 190, there exists a maximum of \( f \) in the interval \([0, \lambda_0]\). This is a contradiction to the supposition that \( y \in I(\omega) \). This proves that \( I(\omega) \subset R(\hat{P}(x)) \setminus \{0\} \).

Let us now prove that the converse, that \( R(\hat{P}(x)) \setminus \{0\} \subset I(\omega) \). Let \( y \in R(\hat{P}(x)) \setminus \{0\} \). If there exists \( \lambda_0 \) such that \( u(\omega + \lambda_0 y) \geq \max_{\lambda \geq 0} u(\omega + \lambda y) \) then by uniformity we have \( u(\omega + \lambda y) \geq u(\omega + \lambda_0 y) \), implying \( u(\omega + \lambda y) \geq u(\omega + \lambda_0 y) \) for all \( \lambda \geq 0 \). This means that there exists a half-line in the indifference surface, a contradiction. Hence \( y \in I(\omega) \). ■

### 4.4 Equivalences of closures of cones

The next Proposition implies that in the frameworks of C (1995,1997), the closures of the global cone, the increasing cone, and the recession cone of the set of allocations weakly preferred to the initial endowment, all coincide.

**Proposition 7.** Assume \( u(\cdot) \) is a quasi-concave and continuous utility function satisfying monotonicity (10) and uniform nonsatiation (19). Then

\[
\overline{A}(\omega) = \overline{T}(\omega) = R(\hat{P}(x)).
\]

**Proof.**

Step 1. Quasi-concavity (A.1) and uniform nonsatiation (19) imply uniformity of recession cones (16), that is, for all \( x \) and \( x' \), it holds that \( R(\hat{P}(x)) = R(\hat{P}(x')) \) (Monteiro, Page and Wooders 2000).

Step 2. Quasi-concavity, (A-1), and (16) imply that \( T(\omega) = R(\hat{P}(x')) \) (Monteiro, Page and Wooders 2000).

Step 3. Monotonicity implies \( i^* R(\hat{P}(\omega)) \neq \emptyset \).

Step 4. Quasi-concavity, global uniformity of arbitrage cones (16) and \( i^* R(\hat{P}(\omega)) \neq \emptyset \) imply \( i^*(R(\hat{P}(\omega))) \subset A(\omega) \) (Dana, Le Van, Magnien 1999).

Step 5. Quasi-concavity and (19) imply \( A(\omega) \subset R(\hat{P}(\omega)) \).

To sum up, \( i^*(R(\hat{P}(\omega))) \subset A(\omega) \subset R(\hat{P}(\omega)) \) implies that \( \overline{A}(\omega) = R(\hat{P}(\omega)) \). ■

**Remark.** (Monteiro, Page and Wooders 2000). Note that if \( u(\cdot) \) satisfies strict quasi-concavity, then \( u(\cdot) \) satisfies (17), no half lines, automatically. Thus, it follows
from Theorem 1 in Monteiro, Page, and Wooders (1997) that if \( u(\cdot) \) is strictly quasi-concave, then

\[
I(x) = R(\hat{P}(x)) \setminus \{0\}, \text{ for all } x \in X.
\]

### 4.5 Uniformity of the increasing cone and the arbitrage cone.

According to our next Proposition, if the agent’s utility function is only quasi-concave but satisfies uniform nonsatiation (19), then \( u(\cdot) \) will again satisfy global uniformity of arbitrage cones (16).

**Proposition 8.** (Monteiro, Page and Wooders). Let \( u(\cdot) : X \to \mathbb{R} \) be a continuous, quasi-concave utility function. If \( u(\cdot) \) satisfies uniform nonsatiation, (19) then \( u(\cdot) \) also satisfies global uniformity of arbitrage cones (16).

**Proof.** First, recall that (19) implies (22a) and (22b) (see Remark 1). Now suppose \( y \in R(\hat{P}(\omega)) \). Then there exist a sequence \( \{x^n\} \subseteq \hat{P}(\omega) \) and \( \lambda^n \downarrow 0 \) such that \( \lambda^n x^n \to y \). If \( x \in X \) is given and \( x^n \in \hat{P}(x) \) for infinitely many \( n \) then \( y \in R(\hat{P}(x)) \).

If for some \( n^0, n \geq n^0 \) implies that \( x^n \not\in \hat{P}(x) \) (i.e., \( u(x^n) < u(x) \)), define \( r = u(\omega) \) and \( s = u(x) \). By (22a), there is a sequence \( \{\bar{x}^n\} \) with \( u(\bar{x}^n) = u(x) \) such that \( |x^n - \bar{x}^n| \leq N(u(x^n), s) \).

By (22b), \( N(\cdot, \cdot) \) is bounded on bounded sets, so that \( \lambda^n N(u(x^n), s) \to 0 \). Thus \( |\lambda^n x^n - \lambda^n \bar{x}^n| \to 0 \) implying that \( \lambda^n \bar{x}^n \to y \). This in turn implies that \( y \in R(\hat{P}(x)) \). Since \( \omega \) and \( x \) are chosen arbitrarily, we conclude that the arbitrage cones are all equal. \( \blacksquare \)

### 4.6 The closed gradient condition

In this section, we show that the closed gradient condition of C (1997), along with concavity of a utility function, imply that the global cone (A), the increasing cone I, the strict increasing cone I^s and the interior of the arbitrage cone int\( R \) are all equal.

Since the utility function is concave, \( R(\hat{P}(x)) = R(\hat{P}(x')) \forall x, x' \in X \). Define

\[
R := R(\hat{P}(x)).
\]

By our maintained assumption that the utility function is increasing, \( \mathbb{R}_+^l \subseteq R \implies \mathcal{i}^\tau(R) \neq \emptyset \).

**Lemma 4.** Assume \( u \) is continuously differentiable and that \( \sup_{x \in X} u(x) = +\infty \). Then \( \mathcal{i}^\tau(\hat{P}(x)) = P(x), \forall x \).

**Proof.** In notes.

**Lemma 5.** Assume \( u \) is concave. If \( y \in \mathcal{i}^\tau(R) \) and \( x \in X \), the function \( \lambda \to u(x + \lambda y) \) is non decreasing and \( \lim_{\lambda \to \infty} u(x + \lambda y) = +\infty \).

**Proof:** See Dana, Le Van and Magnien, (1999, Proposition 2(5)).

**Lemma 6.** Assume \( u \) is concave and continuously differentiable. Let \( y \in R \).
(i) Then $\forall x \in X$, $\nabla u(x) \cdot y \geq 0$.

(ii) If $\nabla u(x) \cdot y = 0$, then $u(x + \lambda y) = u(x), \forall \lambda \geq 0$.

**Proof.**

(i) From Lemma 5 and the concavity of $u$, we have

$$0 \geq u(x) - u(x + \lambda y) \geq -\lambda \nabla u(x) \cdot y, \forall \lambda \geq 0.$$ 

Hence $\forall x \in X$, $\nabla u(x) \cdot y \geq 0$.

(ii) If $\nabla u(x) \cdot y = 0$, then

$$0 \geq u(x) - u(x + \lambda y) \geq -\lambda \nabla u(x) \cdot y = 0$$

and thus $u(x + \lambda y) = u(x), \forall \lambda \geq 0$. ■

**Lemma 7.** Assume $u$ is concave and $\sup_{x \in X} u(x) = +\infty$ (21). Then:

(i) $A(x)$ is independent of $x$ and:

$$A(x) = \left\{ y \in R : \lim_{t \to +\infty} u(x + ty) = +\infty \right\}.$$ 

(ii) $i^\circ(R) \subseteq A(x) \subseteq R$.

**Proof.** See Dana, Le Van and Magnien, JET 99, Proposition 2(3).

We say that $y \in X$ is a half-line direction if there exists $x \in X$, such that $u(x + \lambda y) = u(x), \forall \lambda \geq 0$. Let $HL$ denote the set of half-lines directions.

Observe that the map given in the definition of the closed gradient condition (20) is well-defined: since there exists no saturation point and since the utility function is concave, $\forall x \in X$, $\nabla u(x) \neq 0$.

**Proposition 9.** Assume $u$ is continuously differentiable and concave (21). Then:

(i) $HL \subseteq \partial R$

(ii) If, in addition, the closed gradient condition (20) holds, then $HL = \partial R$.

**Proof.**

(i) Let $y \in HL$. Obviously, $y \in R$. If $y \in i^\circ(R)$ then $\forall x, \lim_{\lambda \to +\infty} u(x + \lambda y) = +\infty$ (Lemma 2): a contradiction. Hence $y \in \partial R$.

(ii) Let $\sigma_\alpha = \left\{ x : u(x) = \alpha \right\}$, where $\alpha \in \mathbb{R}$. We will show that if $y \in \partial R$ then there exists $\tilde{x} \in \sigma_\alpha$ such that $u(\tilde{x} + \lambda y) = u(\tilde{x}) = u(x), \forall \lambda \geq 0$. Obviously, if $y = 0$, the claim is true. So, assume $y \in \partial R \setminus \{0\}$.

Take some $x \in \sigma_\alpha$. If $u(x + \lambda y) = u(x)$ for any $\lambda \geq 0$, the claim is true. So we can assume there exists $\mu > 0$, such that $u(x + \mu y) > u(x)$. If $\mu > 1$, the concavity of $u$ implies that $u(x + y) > u(x)$. If $\mu < 1$, then $u(x + y) \geq u(x + \mu y)$ (Lemma 2) and thus $u(x + y) > u(x)$. This implies that $x + y \in i^\circ(\bar{P}(x))$ (Lemma 1). There
exists a ball $B(x + y, \rho)$ (we choose $\rho < \|y\|$) which is included in $P(x)$. Take a sequence $(x + y^n)$ in $B(x + y, \rho)$ which converges to $x + y$, and $y^n \notin R, \forall n$ (this is possible because $y \in \partial R \setminus \{0\}$). Therefore, for any $n$, there exists $\lambda^n > 1$, such that $u(x + \lambda^n y^n) = u(x)$ and $u(x + \lambda y) < u(x)$ if $\lambda > \lambda^n$.

Define $x^n = x + \lambda^n y^n$. We claim that $\|x^n\| \to +\infty$. Indeed, in the contrary, one can assume $x^n \to z$. Observe that $z \neq x$ (because $\|x^n - x\| \geq \|y^n\|$). We have $z = x + \mu y$, with $\mu \geq 1$, and consequently $u(z) \geq u(x + y) > u(x)$: a contradiction with $u(z) = \lim u(x^n) = u(x)$. We have proved that $\|x^n\| \to +\infty$. Since $y^n \to y$, we have $\lambda^n \to +\infty$.

Now we have by the concavity of $u$:

$$0 = u(x^n) - u(x) \geq \nabla u(x^n) \cdot (x^n - x).$$

This implies:

$$\frac{\nabla u(x^n)}{\|\nabla u(x^n)\|} x^n \leq \frac{\nabla u(x^n)}{\|\nabla u(x^n)\|} x.$$

When $n \to +\infty$, $\frac{x^n}{\|x^n\|} \to y$, $\frac{x}{\|x\|} \to 0$, and by the closed gradient condition,

$$\frac{\nabla u(x^n)}{\|\nabla u(x^n)\|} \to \frac{\nabla u(\hat{x})}{\|\nabla u(\hat{x})\|},$$

where $\hat{x} \in \sigma_{\alpha}$. We obtain that $\frac{\nabla u(\hat{x})}{\|\nabla u(\hat{x})\|} \cdot y \leq 0$. By Lemma 3 (i) we then have

$$\frac{\nabla u(\hat{x})}{\|\nabla u(\hat{x})\|} \cdot y = 0,$$

and by Lemma 3(ii), $u(\hat{x} + \lambda y) = u(\hat{x}), \forall \lambda \geq 0$. The proof is complete. ■

The next result requires, in addition, a part of the uniform nonsatiation condition, specifically that the supremum of the norm of the gradient is bounded.

$$\sup_{x \in X} \|\nabla u(x)\| \leq K. \tag{24}$$

**Lemma 8.** Assume $u$ is continuously differentiable, concave and (24) holds. Also assume $u$ is increasing and the closed gradient condition holds. Let $x \in X$, and $y \in \partial R \setminus \{0\}$. Then $\lim_{\lambda \to +\infty} u(x + \lambda y) < +\infty$.

**Proof.** Assume $\lim_{\lambda \to +\infty} u(x + \lambda y) = +\infty$. From the proof of part (ii) of the previous Proposition there exists $\hat{x} \in \sigma_{\alpha} (\alpha = u(x))$ which satisfies $u(\hat{x} + \lambda y) = u(\hat{x}) = u(x), \forall \lambda \geq 0$. For any $\lambda \geq 0$, by (A5), we have

$$|u(\hat{x} + \lambda y) - u(x + \lambda y)| \leq K \|\hat{x} - x\|$$

and hence

$$u(x + \lambda y) \leq K \|\hat{x} - x\| + u(\hat{x} + \lambda y) = K \|\hat{x} - x\| + u(x).$$

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But $u(x + \lambda y)$ converges to $+\infty$ when $\lambda$ tends to $+\infty$: a contradiction. ■

We want to prove that under the assumptions that $u$ is continuously differentiable, concave and satisfies both the closed gradient condition and uniform non-satiation (19), for any $x \in X$, we have: $G(x) = A(x) = I(x) = i^r(R)$. We require additional lemmas.

**Lemma 9.** Let $y \in \partial R \setminus \{0\}$. Then there exists $(z_1, \ldots, z_{l-1})$ with $z_i \in i^r(R), \forall i = 1, \ldots, l - 1$ and $(z_1, \ldots, z_{l-1}, y)$ are linearly independent.

**Proof.** Since $\mathbb{R}^l_+ \subseteq R$, there exist $(z_1, \ldots, z_{l-1}, z_l)$, with $z_i \in i^r(R), \forall i$, which are linearly independent. Hence one can write $y = \sum_{i=1}^l \alpha_i z_i$, and $\alpha_i \neq 0$ for some $i$. One can assume that $\alpha_l \neq 0$. We claim that $(z_1, \ldots, z_{l-1})$ are linearly independent. If not, we can write $y = \sum_{i=1}^{l-1} \beta_i z_i$ and obtain that $\sum_{i=1}^{l-1} (\alpha_i - \beta_i) z_i + \alpha_l z_l = 0$. This implies $\alpha_l = 0$ contradicting our assumption. ■

**Proposition 10.** Assume $u$ is continuously differentiable, concave and (24) holds. Also assume $u$ is increasing and the closed gradient condition holds. Let $x \in X$, and $y \in \partial R \setminus \{0\}$. Then the function defined on $\mathbb{R}_+: \lambda \to u(x + \lambda y)$ has a maximum.

**Proof.** From Lemma 8, we know that under our assumptions, for any $x \in X$ and for any $y \in \partial R \setminus \{0\}$ it holds that $\lim_{\lambda \to +\infty} u(x + \lambda y) < +\infty$.

Consider the function $f(\lambda) = u(x + \lambda y), \lambda \in \mathbb{R}_+$. Assume that $f$ has no maximum. We have $f'(\lambda) = \nabla u(x + \lambda y) \cdot y$. Since $f$ is concave, its derivative $f'$ is decreasing (or nonincreasing). Let $\theta = \lim_{\lambda \to +\infty} f'(\lambda)$. We have $\theta \geq 0$ by Lemma 6 — $(\nabla u(x + \lambda y) \cdot y > 0)$. If $\theta > 0$, we have $u(x + y) - u(x) = \int_0^\lambda f'(t)dt \geq \theta \lambda$ and hence $\lim_{\lambda \to +\infty} u(x + \lambda y) = +\infty$: a contradiction with Lemma 8 ($\lim \lambda \to +\infty$ is finite). Hence $\theta = 0$, i.e., $\lim_{\lambda \to +\infty} \nabla u(x + \lambda y) \cdot y = 0$.

By the closed gradient condition and uniform nonsatiation, we can assume that when $\lambda \to +\infty$, $\nabla u(x + \lambda y) \to \gamma \neq 0$. Consider the $L$ linearly independent vectors in the statement of Lemma 9, $(z_1, \ldots, z_{l-1}, y)$. Let $\Pi_1$ denote the plane containing $x$ and spanned by $(y, z_1)$. Let $\alpha = \lim_{\lambda \to +\infty} u(x + \lambda y)$. Since $u(x + \lambda z_1) \to +\infty$ when $\lambda \to \infty$, it holds that $\Pi_1 \cap \sigma_\alpha \neq \emptyset$, where $\sigma_\alpha$ is the $\alpha$-indifference surface. Denote $C_\alpha = \Pi_1 \cap \sigma_\alpha$. Using the same argument as in the proof of the statement (ii) of Proposition 9, we show that there exists $\hat{x} \in C_\alpha$ such that

$$u(\hat{x} + \lambda y) = u(\hat{x}) = \alpha, \forall \lambda \geq 0.$$ 

Observe that $\hat{x}$ cannot belong to the line $\{x + \lambda y : \lambda \in R\}$ because $f$ has no maximum on this half-line.

We have $0 \geq u(x + \lambda y) - u(\hat{x} + \lambda y) \geq \nabla u(x + \lambda y) \cdot (x - \hat{x})$. Let $\lambda$ converge to $+\infty$. We obtain:

$$\gamma \cdot (x - \hat{x}) \leq 0$$

because $u(\hat{x} + \lambda y) = \alpha$ and $\lim_{\lambda \to +\infty} u(x + \lambda y) = \alpha$.

One can easily check that $\nabla u(\hat{x}) \cdot y = 0$. Write $\zeta = \nabla u(\hat{x})$. Let $\zeta_1, \gamma_1$ respectively denote the orthogonal projections of $\zeta$ and $\gamma$ on $\Pi_1$. We have either $\gamma_1 = \mu \zeta_1$ or $\gamma_1 = \zeta_1$.
$-\mu \zeta_1$, with $\mu \geq 0$ (since two vectors in $\Pi_1$ that are orthogonal to the same vector of $\Pi_1$ must be proportional). Since $z_1 \in i^*(R)$, for any $\lambda \geq 0$ we have $\nabla u(x + \lambda y) \cdot z_1 > 0$ which implies that $\gamma_1 \cdot z_1 \geq 0$. But we have also $\nabla u(\bar{x}) \cdot z_1 > 0 \implies \zeta_1 \cdot z_1 > 0$. These inequalities imply $\gamma_1 = \mu \zeta_1$ with $\mu \geq 0$.

We have

$$u(\bar{x}) - u(x + \lambda y) \\ \geq \nabla u(\bar{x}) \cdot (\bar{x} - x) - \lambda \nabla u(\bar{x}) \cdot y \\ = \nabla u(\bar{x}) \cdot (\bar{x} - x) = \zeta_1 \cdot (\bar{x} - x)$$

from concavity.

Let $\lambda \to \infty$. We obtain $\zeta_1 \cdot (\bar{x} - x) \leq 0$. Remember that $\gamma_1 \cdot (x - \bar{x}) \leq 0$.

We claim that the orthogonal projection $\gamma_1$ of $\gamma$ is equal to zero. If not we have that $\gamma_1 \cdot (x - \bar{x}) = 0$, because $\gamma_1 = \mu \zeta_1$ with $\mu \geq 0$. But $\gamma_1 \cdot y = 0$. Since $y$ and $(x - \bar{x})$ are linearly independent, we have $\gamma_1 = 0$, i.e., $\gamma$ is orthogonal to $\Pi_1$.

By the same way, we prove that $\gamma$ is orthogonal to $\Pi_i, i = 2, \ldots, L - 1$. In other words, $\gamma$ is orthogonal to $(z_1, \ldots, z_{L-1}, y)$. Hence $\gamma = 0$. That contradicts that $\gamma \neq 0$.

One must conclude that the function $f$ has a maximum. ■

As a corollary, we have:

**Theorem 1.** Assume that $u$ is continuously differentiable, concave and satisfies both the closed gradient condition (20) and uniform non-satiation (19). Then for any $x$,

$$A(x) = G(x) = I(x) = i^* R.$$ 

**Proof.** From Proposition 6, we have that $I(x) = i^* R$. From Proposition 1, $G(x) = I(x)$. From Lemma 7, $i^* R \subseteq A(x)$. By Lemma 8, $\partial R \setminus \{0\} = HL$. From the definition of $A(x)$ and Proposition 10 it holds that $HL \cap A(x) = \emptyset$. Thus, $A(x) \subseteq i^* R$ and the proof is complete. ■

## 5 Existence of equilibrium

With the above results now in hand, we can proceed to study existence of equilibrium. First, some definitions are required.

Given prices $p \in B := \{p' \in \mathbb{R}^L : \|p'\| \leq 1\}$ the budget set for the $j$th agent is given by $B(\omega_j, p) = \{x \in X_j : \langle x, p \rangle \leq \langle \omega_j, p \rangle\}$. An *equilibrium* for the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ is an $(n + 1)$ -tuple of vectors $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ such that:

(i) $(\bar{x}_1, \ldots, \bar{x}_n) \in A(\omega)$;

(ii) $\bar{p} \in B \setminus \{0\}$; and (iii) for each $j$, $\langle \bar{x}_j, \bar{p} \rangle = \langle \omega_j, \bar{p} \rangle$ and $(iii) P_j(\bar{x}_j) \cap B(\omega_j, \bar{p}) \neq \emptyset$.

An economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage if:

(2.1) whenever $\sum_{j=1}^n y_j = 0$ and $y_j \in R(P_j(\omega_j))$ for all agents $j$ it holds that $y_j = 0$ for all agents $j$. 

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This is the condition of Page (1987), specialized to the case of non-price dependent preferences and of Page and Wooders (1993,1996a,1996b). Page and Wooders (1993,1996a,b) show that no unbounded arbitrage is sufficient for existence of equilibrium and, provided at most one agent has half-lines in his indifference surfaces, the condition is necessary and sufficient. Under the same condition, Page and Wooders (1993,1996a,b) show that no unbounded arbitrage is necessary and sufficient for nonemptiness of the core and, in fact, necessary and sufficient for nonemptiness of the partnered core. Indeed, Page and Wooders (1996b) shows that no unbounded arbitrage necessary and sufficient for existence of a Pareto optimal point.

An economy \((X_j, \omega_j, P_j(\cdot))_{j=1}^n\) satisfies Werner’s condition if

\[ \cap_j \Lambda^+(R_j(P(\omega_j)) \setminus L(P(\omega_j)) \neq \emptyset. \]

Werner’s result, that his condition is sufficient for existence of equilibrium, is quite remarkable and extremely subtle and insightful. In particular, Werner does not require any nonsatiation condition except that there exist useful trades. Werner also notes that under the NHL condition, his condition is necessary and sufficient.

An economy satisfies C’s (1995) and prior condition of limited arbitrage if

\[ \cap_j \Lambda^+(A_j(\omega_j)) \neq \emptyset. \] (25)

The counterexample in Monteiro, Page and Wooders (1997) shows that this condition is inadequate for existence of equilibrium. Variations of this condition appeared in several papers. In C (1997) the condition is LA97,

\[ \cap_j \Lambda^+(I_j(\omega_j)) \neq \emptyset. \] (26)

The difference between the two conditions above is crucial. As we discuss below, the condition (25) is inadequate for existence of equilibrium, even in the NHL case. Within the context of the special assumptions made by C (1995,1997), the condition based on the increasing cone (26) in the NHL case is simply Werner’s (1997) condition. This follows from Proposition 6 and the fact that C’s assumptions imply that the lineality space of an agent’s preferred set is the zero vector.

Thus, the remaining question is whether, in the HL case, with the uniform nonsatiation condition, do either of C’s conditions, (25) or (26), imply existence of equilibrium?

In the remainder of this section we identify a crucial gap in the purported proof in C (1997). We then discuss the problem of existence of equilibrium and the unresolved difficulty with Chichilnisky’s condition. We conclude the section by introducing two different (albeit closely related) conditions that, along with Chichilnisky’s other conditions, allow a proof of existence of equilibrium.

We ask the reader to turn to page 464 of C (1997). The author writes that, under the closed gradient condition (the HL case),

“(a) the global cone \(G_h\) is open (Chichilnisky 1995a) so that its complement \(G_h^c\) is closed, and the set of directions in \(G_h^c\) is compact. On each direction of \(G_h^c\) the utility \(u_h\) achieves a maximum; therefore there exists for each \(h\) a maximum utility level for \(u_h\) over all directions in \(G_h^c\).”
To see the problem with the above claim, consider the following simple analogy. Let the function \( f(x) \) be defined on the closed interval \([0, 1]\) where

\[
f(x) = \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

This function satisfies the conditions that the domain is compact, for each \( x \) in the domain, the value of the function is bounded, yet there does not exist a real number \( k \) such that \( f(x) \leq k \) for all \( x \). If the function were continuous, or upper hemi-continuous, then compactness of the domain would imply the existence of a uniform bound on the value of the function. Chichilnisky does not, however, demonstrate any continuity conditions on the maximum utility level achieved along any ray in the cone. Unfortunately, we have not been able to prove Chichilnisky’s claims under her assumptions.

That there is no proof for C’s claims motivates us to introduce additional assumptions under which her claimed results will hold in the HL case. The simplest of these is that the global cone is a polyhedral cone, that is, it is finitely generated.

**Theorem 2.** Assume that for all agents \( j = 1, \ldots, n \), \( u_j \) is concave (21), continuously differentiable, monotonic (10) and satisfies the closed gradient condition (20). Also assume that for each agent \( j \), the set \( I(\omega_j) \) is a polyhedral cone. Assume the 1997 version of C’s condition (26). Then the economy has an equilibrium.

**Proof.** From (26) it follows that all the sets \( I(\omega_j) \) lie on one side of some hyperplane, given by the vector \( p^* \) and a constant \( a \). The boundaries of some of the sets \( I(\omega_j) \), however, may lie on the hyperplane. But these boundaries are determined by half-lines in indifference curves. Since the cones \( I(\omega_j) \) are polyhedral, they are generated by a finite number of directions, coinciding with half-lines in indifference curves. Thus, utilities in these directions are bounded. It follows that the utilities possibilities set for the economy is bounded. Existence of equilibrium then follows from Allouch (200X).

6 **Examples**

6.1 **Example 1: The increasing cone is not invariant with respect to its origin**

We begin with an example that can be described simply in pictures. In this example the agent’s utility function is continuous and quasi-concave and the agent’s choice set \( X \) is given by \( \mathbb{R}^2 \). Thus (A-1) and (A-2) hold. However, global uniformity of increasing cones (A-7) fails to hold.

There are two commodities and the agent has Leontief preferences with indifference curves kinked along the curve

\[ x_2 = -e^{-x_1}. \]
The Figures 1-4 below summarize the situation:

**Figure 1**
Utility Surface

**Figure 2**
Indifferences Curves

**Figure 3**
\[ I(\bar{x}) = \{ (y_1, y_2) : y_1 > 0 \text{ and } y_2 > 0 \} \]
when \( \bar{x} = (\bar{x}_1, \bar{x}_2) \) with \( \bar{x}_2 < 0 \)

**Figure 4**
\[ I(\bar{x}) = \{ (y_1, y_2) : y_1 > 0 \text{ and } y_2 \geq 0 \} \]
when \( \bar{x} = (\bar{x}_1, \bar{x}_2) \) with \( \bar{x}_2 \geq 0 \)

It can be easily verified that for this example, the increasing cone \( I(\bar{x}) \) and the global cone \( A(\bar{x}) \) are equivalent.

### 6.2 Example 2. The increasing cone may properly contain the global cone

Let \( \alpha \) and \( \beta \) be two functions \( \alpha, \beta : \mathbb{R} \to \mathbb{R} \) such that \( \beta(X) > \alpha(X) > 0 \), \( \beta^0(X) > 0 > \alpha^0(X) \) for every \( x \in \mathbb{R} \).

The agent’s utility function, \( u(\cdot) : \mathbb{R}^2 \to \mathbb{R} \), is defined by

\[
u(x, y) = \begin{cases} 
X & \text{if } (X - x)\beta(X) = y \text{ and } y \geq 0 \\
X & \text{if } (X - x)\alpha(X) = y \text{ and } y < 0
\end{cases}
\]

\( ^6 \) For example take \( \alpha(X) = -\arctan(X) + \pi/2 \) and \( \beta(X) = \arctan(X) + 3\pi/2 \). Thus: \( \alpha^0(X) = -\frac{1}{1+X^2} < 0 < \frac{1}{1+X^2} = \beta^0(X) \) and \( \beta(X) > -\pi/2 + 3\pi/2 = \pi > -\arctan(X) + \pi/2 > 0. \)

20
The function \( u(\cdot) \) is differentiable for \( y \neq 0 \), with indifferece curves given by two half lines intersecting at \((X,0)\) where \( X \) is the utility level. For a fixed \( X \) the half-lines are
\[
\{(x,y) \in \mathbb{R}^2; y \geq 0, y = (X-x)\beta(X)\},
\{(x,y) \in \mathbb{R}^2; y < 0, y = (X-x)\alpha(X)\}.
\]

Hence the utility function satisfies the closed gradient condition, (A-13). Moreover, the utility function is quasi-concave and strictly monotonic.

First, we show that in this example, as the above, the increasing cones are not globally uniform (i.e., (A-7) does not hold).

**Proposition 1** The increasing cone \( I(\omega) \) is not invariant with respect to changes in endowment \( \omega \).

**Proof.** It suffices to consider \( \omega = (a,0) \). Recall that
\[
I(\omega) = \{ v \in \mathbb{R}^2 \setminus \{0\}; \beta(\omega + rv) \}\text{max} r \geq 0.
\]

Suppose first \( v = (v_1,v_2), v_1 \leq 0 \). If \( v_2 \leq 0, u((a,0) + r(v_1,v_2)) \leq u(a,0) = a \) and there is a maximum. If \( v_2 > 0, u((a,0) + r(v_1,v_2)) = u(a + rv_1,rv_2) = X \) where \( rv_2 = (X-a-rv_1)\beta(X) \). Hence \( r = \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} \). If \( v_1 = 0 \) then \( X \to \infty \) if \( r \to \infty \). Now suppose \( v_1 < 0 \). If \( \beta(a) \geq -\frac{v_2}{v_1} \), then for every \( X > a \), \( \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} < 0 \). Hence there is a maximum whenever \( \beta(a) \geq -\frac{v_2}{v_1} \). Finally suppose \( \beta(a) < -\frac{v_2}{v_1} \). Then for every \( X > a \), such that \( \beta(X) < \frac{v_2}{v_1} \), the function \( \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} \) is increasing and positive. If \( X_0 \) is such that \( \beta(X_0) = \frac{v_2}{v_1}\), then since \( u > a \), \( \lim_{X \to X_0} \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} = \infty \). And if \( \beta(X) < -\frac{v_2}{v_1} \) for every \( X \) then \( \lim_{X \to \infty} \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} = \infty \). Thus
\[
I(\omega) = \{ v \in \mathbb{R}^2; v_1 < 0, \beta(a) < -\frac{v_2}{v_1}\} \cup \{ v \in \mathbb{R}^2; v_1 \geq 0, -v_2 < v_1\alpha(a)\}. \quad (27)
\]

Our next Proposition shows that for this example, even though the closed gradient condition, (A-13), is satisfied, the increasing cone may properly contain the closure of the global cone. Thus, the increasing cone and the global cone can differ significantly, depending on the assumptions of the economic model.

**Proposition 2** The increasing cone \( I(\omega) \) may properly contain the closure of the global cone, \( \overline{A(\omega)} \).

**Proof.** It suffices to demonstrate this for \( \omega = 0 \). Also, to simplify the proof we suppose \( \alpha(\infty) = 0 \). Suppose \( v \in \mathbb{R}^2 \setminus \{0\} \). If \( v_2 = 0 \) and \( r > 0 \), we have \( u(rv_1,0) = rv_1 \to \infty \) if \( v_1 > 0 \). If \( v_1 = 0 \) and \( v_2 > 0 \) then \( u(0,rv_2) = X \) where \( X\beta(X) = rv_2 \). Hence if \( r \to \infty, X \to \infty \) and therefore \( v \in A_u(0) \). Finally suppose \( v_1 > 0 > v_2 \). Then \( u(rv) = X \), where \( (X-rv_1)\alpha(X) = rv_2 \). So we have that \( \frac{X_\alpha(X)}{v_1\alpha(X) + v_2} = r \). Now since
the denominator is negative if $X$ is large we have that $u(rv)$ is bounded. Hence we have that $A(0) = \mathbb{R}_+^2 \setminus \{0\}$. By comparison of this expression with $27$ it is clear that $I(0) \supseteq \overline{A(0)}$. $\blacksquare$

To make the example more concrete, let

$$
\begin{align*}
\beta(X) &= \arctan(X) + \frac{\pi}{2}, \\
\alpha(X) &= -\arctan(X) + \frac{3\pi}{2}.
\end{align*}
$$

With these specifications for $\beta(\cdot)$ and $\alpha(\cdot)$, the indifference curve corresponding to any utility level $X$ is given by

$$
y = \begin{cases}
\quad (X - x) \cdot \left(\arctan(X) + \frac{3\pi}{2}\right) & \text{if } x \leq X \\
\quad (X - x) \cdot \left(-\arctan(X) + \frac{\pi}{2}\right) & \text{if } x > X.
\end{cases}
$$

Figure 5 below depicts the indifference curves corresponding to utility levels $X = 1$, $X = 2$, $X = 3$, and $X = 4$.

6.3 Example 3. A differentiable example in the spirit of example 2 above

Define $b : \mathbb{R} \to \mathbb{R}$, by $b(a) = a + \beta(a) - \alpha(a)$. Define $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$
H(x, a) = \begin{cases}
\quad \frac{\beta^2(a) - \alpha^2(a)}{2} - \beta(a)(x - a) & \text{if } x \geq a \\
\quad \frac{\beta^2(a) - \alpha^2(b)}{2} - \beta(a)(x - a) + \frac{(x-a)^2}{2} & \text{if } a \leq x \leq b(a) \\
\quad -\alpha(a)(x - b(a)) & \text{if } b(a) \leq x.
\end{cases}
$$

First note that $H$ is a continuous function. For every $x$, $H(x, \cdot)$ is strictly increasing and onto $\mathbb{R}$. We finally define the utility function, $u : \mathbb{R}^2 \to \mathbb{R}$ implicitly by $H(x, u(x, y)) = y$. To prove that $u$ is continuously differentiable it will be enough, using the implicit function theorem to check that $H$ is continuously differentiable.
This we now proceed to check. It suﬃces to show that \( H \) has continuous partial derivative \( \frac{\partial H}{\partial x}(x, a) \) and \( \frac{\partial H}{\partial a}(x, a) \) for \((x, a) \in \mathbb{R}^2\). Thus

\[
\frac{\partial H}{\partial x}(x, a) = \begin{cases} 
-\beta(a) & \text{if } x \leq a \\
-\beta(a) + x - a & \text{if } b(a) \geq x \geq a \\
-\alpha(a)(x - b(a)) & \text{if } b(a) \leq x.
\end{cases}
\]

and

\[
\frac{\partial H}{\partial a}(x, a) = \begin{cases} 
\beta(a)\beta_0(a) - \alpha(a)\alpha_0(a) - \beta_0(a)(x - a) + \beta(a) & \text{if } x \leq a \\
\beta(a)\beta_0(a) - \alpha(a)\alpha_0(a) - \beta_0(a)(x - a) + \beta(a) - (x - a) & \text{if } b(a) \geq x \geq a \\
-\alpha_0(a)(x - b(a)) + \alpha(a)b_0(a) & \text{if } b(a) \leq x.
\end{cases}
\]

are continuous and hence \( H \) is continuously diﬀerentiable. Since

\[
\frac{\partial H}{\partial x}(x, a) < 0 < \frac{\partial H}{\partial a}(x, a)
\]

it follows that

\[
\frac{\partial u}{\partial x} > 0 \text{ and } \frac{\partial u}{\partial y} > 0.
\]

Since, in the no-half-lines case, this condition is dual to the condition of Page and Wooders (1993,1996a,b), it follows from their results that an equilibrium exists. See Monteiro, Page and Wooders (1999) for references, further discussion, and a counterexample to a proposition on which C bases her claimed existence result using the above condition.

Since the increasing cone \( I_j(\omega_j) \) and the global cone \( A_j(\omega_j) \) can diﬀer depending on the economic model (see examples 2 and 3 above), the conditions \( \% \) and \( \% \) can diﬀer. As remarked above, the failure of C’s condition is due to the fact that the global cone may be too small. This is illustrated by the following example, taken from Monteiro, Page and Wooders (1997). For this example, the global cone of agent 1 \( A_1(\omega_1) \) is depicted in Figure 6 and the increasing cone \( I_1(\omega) \) is depicted in Figure 7.

Figure 6

Figure 7
Here the global cone $A_1(\omega_1)$ is given by the positive orthant, while the increasing cone $I_1(\omega_1)$ is given by the non-negative orthant minus the origin. A specific utility function with global cone and increasing cone depicted in Figures 5 and 6 is given by

$$u_1(x_1, x_2) = x_1 + x_2 + 2 - \sqrt{(x_1 - x_2)^2 + 4}.$$ 

We refer the reader to Monteiro, Page and Wooders (1997) for details.

### 6.4 Noncompactness of the utility possibilities set

Another question that arises in the literature on arbitrage is whether the set of utility possibilities is closed (see Dana, Le Van and Magnien (1996) for informative results and references). The following example shows that the condition $\cap_j D^+(A_j(\omega_j)) \neq \emptyset$ does not imply compactness of the set of utility possibility set.

**Example** Consider the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^2$ with $X_j = \mathbb{R}^2$ and $\omega_i = 0$ for $i = 1, 2$. Agent one’s preference are given by the utility function

$$u_1(x, y) = \begin{cases} x_1 & \text{if } x_1 \leq 0 \text{ or } x_2 \geq -1 \\ -\frac{x_1}{x_2} & \text{if } x_1 \geq 0 \text{ and } x_2 \leq -1, \end{cases}$$

while agent two has utility function given by

$$u_2(x_1, x_2) = x_1 + 2x_2.$$

Figure 8 below summarizes the model.

![Figure 8](image.png)

The increasing cones are

$$I_1(0) = \{v \in \mathbb{R}^2; v_1 > 0, v_2 \geq 0\}$$
$$I_2(0) = \{v \in \mathbb{R}^2; v_1 + 2v_2 > 0\}.$$ 

Note that

$$(1, 2) \in D^+(I_1(0)) \cap D^+(I_2(0)).$$
Thus, the condition $\bigcap_j D^+(I_j(\omega_j)) \neq \emptyset$ holds. The set of utility possibilities, however, is not compact. To see this, consider the utility allocation $(u_1(-X_n), u_2(X_n))$ where $X_n = (-n, n)$. Then $u_2(X_n) = -n + 2n = n \to \infty$ and $u_1(-X_n) = \frac{-n}{n} = 1 > 0$. Even though the set of utility possibilities is not compact, this economy has an equilibrium with price $p = (1, 2)$ and allocation $X_1 = (2, -1), X_2 = (-2, 1)$.

**Remark 3** The utility function $u_1$ also does not have an increasing cone satisfying global uniformity. For example $I_1((1/2, -1/2) = \{v \in \mathbb{R}^2; v_1 > 0, v_2 > -v_1\}$.

**References**


