Modeling continuous-time financial markets with capital gains taxes

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Abstract

We formulate a model of continuous-time financial market with risky asset subject to capital gains taxes. We study the problem of maximizing expected utility of future consumption within this model both in the finite and infinite horizon. Our main result is that the maximal utility does not depend on the taxation rule. This is shown by exhibiting maximizing strategies which tracks the classical Merton optimal strategy in tax-free financial markets. Hence, optimal investors can avoid the payment of taxes by suitable strategies, and there is no way to benefit from tax credits.

Key Words and phrases: Optimal consumption and investment in continuous-time, capital gains taxes.

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1 Introduction

Since the seminal papers of Merton [6, 7], there has been a continuous interest in the theory of optimal consumption and investment decision in financial markets. A large literature has focused particularly on the effect of market imperfections on the optimal consumption and investment decision, see e.g. Cox and Huang [1] and Karatzas, Lehoczky and Shreve [5] for incomplete markets, Cvitanić and Karatzas [2] for markets with portfolio constraints, Davis and Norman [3] for markets with proportional transaction costs.

However, there is a very limited literature on the capital gains taxes which apply to financial securities and represent a much higher percentage than transaction costs. Compared to ordinary income, capital gains are taxed only when the investor sells the security, allowing for a deferral option. One may think that the taxes on capital gains have an appreciable impact on individuals consumption and investment decisions. Indeed, under taxation of capital gains, an investor supports supplementary charges when he rebalances his portfolio, which alters the available wealth for future consumption, possibly depreciating consumption opportunities compared to a tax-free market. On the other hand, since taxes are paid only when embedded capital gains are actually realized, the investor may choose to defer the realization of capital gains and liquidate his position in case of capital losses, particularly when the tax code allows for tax credits. Previous works attempted to characterize intertemporal consumption and investment decisions of investors who have permanently to choose between two conflicting issues : realize the transfers needs for an optimally diversified portfolio, or use the ability to defer capital gains taxes.

The taxation code specifies the basis to which the price of a security has to be compared in order to evaluate the capital gains (or losses). The tax basis is either defined as (i) the specific share purchase price, or (ii) the weighted average of past purchase prices. In some countries, investors can chose either one of the above definitions of the tax basis. A deterministic model with the above definition (i) of the tax basis, together with the first in first out rule of priority for the stock to be sold, has been introduced and studied by Jouini, Koehl and Touzi [9, 10].

The case where the tax basis is defined as the weighted average of past purchase prices is easier to analyze, as the tax basis can be described by a controlled Markov dynamics. Therefore, it can be treated as an additional state variable in a classical stochastic control problem. A discrete-time formulation of this model with short sales constraints and linear taxation rule has been studied by Dammon, Spatt and Zhang [4]. They considered the problem of maximizing the expected discounted utility of future consumption, and provided a numerical analysis of this model based on the dynamic programming principle. In particular, they showed that investors may optimally sell assets with embedded capital gains, and that the Merton tax-free optimal strategy is approximately optimal for ”young investors”. We refer to Gallmeyer, Kaniel and Tompaidis [11] for an extension of this analysis to the multi-asset framework.

The first contribution of this paper is to provide a continuous-time formulation of the utility maximization problem under capital gains taxes, see Section 2. The financial market consists of a tax exempt riskless asset and a risky one. Transfers are not subject to any
transaction costs. The holdings in risky assets are subject to the no-short sales constraints, and the total wealth is restricted by the no-bankruptcy condition. The risky asset is subject to taxes on capital gains. The tax basis is defined as the weighted average of past purchase prices. We also introduce a possible fixed delay in the tax basis. In contrast with [4], we consider a general nonlinear taxation rule. Our results hold both for finite and infinite horizon models. Section 3 shows that the reduction of our model to the tax-free case produces the same indirect utility than the classical Merton model.

The main result of our paper states that the value function of the continuous-time utility maximization problem with capital gains taxes coincides with the Merton tax-free value function. In other words, investors can optimally avoid taxes and realize the same indirect utility as in the tax-free market. We also provide a maximizing strategy which shows how taxes can be avoided. This result is first proved in Section 4 in the case where no tax credits are allowed by the taxation rule. It is then extended to general taxation rules in Section 5 by reducing the problem to the linear taxation rule. The particular tractability of the linear taxation rule case allows to prove that it is optimal to take advantage of the tax credits by realizing immediately capital losses.

From an economic viewpoint, our result shows that capital gains taxes do not induce any tax payment by optimal investors. This suggests that the incorporation of capital gains taxes in financial market models should be accompanied by another market imperfection, as transaction costs, which prevents optimal investors from implementing the maximizing strategies exhibited in this paper. This aspect is left for future research.

2 The Model

2.1 The Financial Market

We consider a financial market consisting of one bank account with constant interest rate \( r > 0 \), and one risky asset with price process evolving according to the Black and Scholes model:

\[
dS_t = \mu S_t dt + \sigma S_t dW_t,
\]

where \( \mu \) is a constant instantaneous mean rate of return, \( \sigma > 0 \) is a constant volatility parameter, and the process \( W = \{W_t, 0 \leq t\} \) is a standard Brownian motion defined on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

Let \( \mathbb{F} \) be the \( \mathbb{F} \)-completion of the natural filtration of the Brownian motion. In order for positive investment in the risky asset to be interesting, we shall assume throughout that

\[
\mu > r.
\]

We also assume that the financial market is not subject to any transaction costs, and the shares of the stock are infinitely divisible.

2.2 Relative Tax Basis

The sales of the stock are subject to taxes on capital gains. The amount of tax to be paid for each sale of risky asset is computed by comparison of the current price to the weighted
average of price of the assets in the investor portfolio. We therefore introduce the relative tax basis process \( B_t \) which records the ratio of the weighted average price of the assets in the investor portfolio to the current price. When \( B_t \) is less than 1, the current price of the risky asset is greater than the weighted-average purchase price of the investor so if she sells the risky asset, she would realize a capital gain. Similarly, when \( B_t \) is larger then 1, the sale of the risky asset corresponds to the realization of a capital loss.

**Example 2.1** Let \( 0 \leq t_0 < t_1 < t_2 < t_3 \) be some given trading dates, and consider the following discrete portfolio strategy:
- buy 5 units of risky asset at time \( t_0 \),
- sell 1 unit of risky asset at time \( t_1 \),
- buy 2 units of risky assets at time \( t_2 \),
- sell 4 units of risky asset at time \( t_3 \),
- buy 2 units of risky assets at time \( t_3 \).

The relative tax basis is not defined strictly before the first purchase date \( t_0 \), and is equal to one exactly at \( t_0 \). We set by convention

\[
B_t = 1 \quad \text{for} \quad t \leq t_0.
\]

Sales do not alter the basis. Therefore, we only care about purchases in order to determine the basis at each time. At times \( t_2 \) and \( t_4 \), the relative tax basis is given by

\[
B_{t_2} = \frac{5S_{t_0} + 2S_{t_2}}{7S_{t_2}} \quad \text{and} \quad B_{t_3} = \frac{5S_{t_0} + 2S_{t_2} + 2S_{t_3}}{9S_{t_3}}.
\]

Although no purchases occur in the time intervals \((t_0, t_2)\), \((t_2, t_3)\), the relative tax basis moves because of the change of the current price:

\[
(BS)_t = \begin{cases} 
(BS)_{t_0} & \text{for} \quad t_0 \leq t < t_2, \\
(BS)_{t_2} & \text{for} \quad t_2 \leq t < t_3 \\
(BS)_{t_3} & \text{for} \quad t \geq t_3.
\end{cases}
\]

### 2.3 Taxation rule

Each monetary unit of stock sold at some time \( t \) is subject to the payment of an amount of tax computed according to the relative tax basis observed at the prior time

\[
t_\delta := (t-\delta)^+ = \max\{0, t-\delta\}.
\]

Here \( \delta \geq 0 \) is a fixed characteristic of the taxation rule. Another characteristic of the taxation rule is the amount of tax to be paid per unit of sale. This is defined by

\[
f(B_{t_3}) ,
\]

where \( f \) is a map from \( \mathbb{R}_+ \) into \( \mathbb{R} \) satisfying

\[
f \text{ non-increasing, } f(1) = 0, \quad \text{and} \quad \liminf_{b \uparrow 1} \frac{f(b)}{b-1} > -\infty.
\]
Example 2.2 (Proportional tax on non-negative gains) Let
\[ f(b) := \alpha (1 - b)^+ \quad \text{for some constant} \quad 0 < \alpha < 1. \]
When the relative tax basis is less than unity, the investor realizes a capital gain, and pays the amount of tax \( \alpha (1 - B_t) \) per unit amount of sales.

Example 2.3 (Proportional tax with tax credits) Let
\[ f(b) := \alpha (1 - b) \quad \text{for some constant} \quad 0 < \alpha < 1. \]
When the relative tax basis is less than unity, the investor realizes a capital gain, and pays the amount of tax \( \alpha (1 - B_t) \) per unit amount of sales. When the relative tax basis is larger then unity, the investor receives the tax credit \( \alpha (B_t - 1) \) per unit amount of sales.

Remark 2.1 When there are no tax credits, i.e. \( f \geq 0 \), it is clear that the total tax paid by the investor is non-negative, and the investor can not do better than in a tax-free market. However, when \( f \) is not non-negative, it is not obvious that the investor can not take advantage of the tax credits, and do better than in a tax-free model. Of course, this would not be acceptable from the economic viewpoint. Our analysis of this situation in Section 5 shows that the presence of tax credits does not produce such a non-desirable effect.

2.4 Investment-consumption strategies
An investor start trading at time \( t = 0 \) with an initial capital \( x \) in cash and \( y \) monetary units in the risky asset. At each time \( t \geq 0 \), trading occurs by means of transfers between the two investment opportunities.

We denote by \( L := (L_t, t \geq 0) \) the process of cumulative transfers form the bank account to the risky assets one, and \( M := (M_t, t \geq 0) \) the process of cumulative transfers from the risky assets account to the bank. Here, \( L \) and \( M \) are two \( \mathcal{F} \)-adapted, right-continuous, non-decreasing processes with \( L_{0-} = M_{0-} = 0 \).

In addition to the trading activity, the investor consumes in continuous-time at the rate \( C = \{C_t, t \geq 0\} \). The process \( C \) is \( \mathcal{F} \)-adapted and nonnegative.

Given a consumption-investment strategy \( (C, L, M) \), we denote by \( X_t \) the position on the bank, \( Y_t \) the position on the risky assets account, and \( B_t \) the relative tax basis at time \( t \). We also introduce the total wealth
\[ Z_t := X_t + Y_t, \quad t \geq 0. \]

2.5 Portfolio constraints
We first restrict the strategies to satisfy the no-bankruptcy condition
\[ Z_t \geq 0 \quad \text{\( \mathcal{F} \)-a.s. for all} \quad t \geq 0, \quad (2.6) \]
i.e. the total wealth of the portfolio at each time has to be non-negative. We also impose the no-short sales constraint
\[ Y_t \geq 0 \quad \mathbb{P} \text{-a.s.} \quad \text{for all} \quad t \geq 0, \tag{2.7} \]
together with the absorption condition
\[ Y_{t_0} (\omega) = 0 \text{ for some } t_0 \implies Y_t (\omega) = 0 \text{ for a.e. } \omega \in \Omega. \tag{2.8} \]
The latter is a technical condition which is needed for a rigorous continuous-time formulation of our problem.

The consumption-investment strategy \((C, \tilde{L}, \tilde{M})\) is said to be \textit{admissible} if the resulting state variables \((X, Y, B)\) satisfy the above conditions (2.6)-(2.7)-(2.8). In particular, the process \((Z, Y)\) is valued in the closure \(\bar{S}\) of the subset of \(\mathbb{R}^2\):
\[ S := (0, \infty) \times (0, \infty). \tag{2.9} \]

Finally, given an admissible strategy \((C, \tilde{L}, \tilde{M})\), we introduce the stopping time:
\[ \tau := \inf \{ t \geq 0 : Y_t \not\in (0, \infty) \} = \inf \{ t \geq 0 : Y_t = 0 \}, \]
where the last equality follows from (2.7). In view of (2.8), it is clear that the trading strategy can be described by means of the non-decreasing right-continuous processes \((L_t, M_t)_{t \geq 0}\) which are related to \((\tilde{L}_t, \tilde{M}_t)_{t \geq 0}\) by
\[ L_t := \int_0^t Y_t^{-1} d\tilde{L}_t \quad \text{and} \quad M_t := \int_0^t Y_t^{-1} d\tilde{M}_t, \quad t < \tau. \]
Here, \(dL_t\) and \(dM_t\) represent the proportion of transfers of risky assets.

### 2.6 Controlled dynamics

Let \((C, \tilde{L}, \tilde{M})\) be an admissible strategy, and define \((L, M)\) as in the previous paragraph. We shall denote \(\nu := (C, L, M)\), and \((X^\nu, Y^\nu, B^\nu) := (X, Y, B)\) the corresponding state variables.

Given an initial capital \(x\) on the bank account, the evolution of the wealth on this account is described by the dynamics:
\[ dX^\nu_t = (rX^\nu_t - C_t)dt - Y^\nu_t dL_t + Y^\nu_t \left(1 - f(B^\nu_{t-})\right) dM_t \quad \text{and} \quad X_0^\nu = x, \tag{2.10} \]
recall from (2.3) that \(t_\delta := (t - \delta)^+.\) Given an initial endowment \(y\) on the risky assets account, the evolution of the wealth on this account is also clearly given by
\[ dY^\nu_t = Y^\nu_t \left( \frac{dS_t}{S_t} + dL_t - dM_t \right) \quad \text{and} \quad Y_0^\nu = y. \tag{2.11} \]
This implies that the total wealth evolves according to
\[ dZ^\nu_t = r(Z^\nu_t - C_t)dt + Y^\nu_t [(\mu - r)dt + \sigma dW_t] - Y^\nu_t f(B^\nu_{t-}) dM_t \quad \text{and} \quad Z_0^\nu = y + x. \tag{2.12} \]
In order to specify the dynamics of the relative tax-basis, we introduce the auxiliary process $K^\nu := B^\nu Y^\nu$. By definition of $B^\nu$, we have:

\[
dK^\nu_t = Y^\nu_t dL_t - K^\nu_t dM_t \quad \text{and} \quad K^\nu_0 = y,
\]

since $B^\nu_0 = 1$. Observe that the contribution of the sales in the dynamics of $K_t$ is evaluated at the basis price. We then define the relative basis process $B^\nu$ by

\[
B^\nu_t = \begin{cases} 
1 & \{Y^\nu_t = 0\} \\
\frac{K^\nu_t}{Y^\nu_t} & \{Y^\nu_t \neq 0\}
\end{cases}.
\]

Hence, the position of the investor resulting from the strategy $\nu$ is described by the triple $(Z^\nu, Y^\nu, B^\nu)$. We call $(Z^\nu, Y^\nu, B^\nu)$ the state process associated with the control $\nu$.

**Proposition 2.1** Let $\nu = (C, L, M)$ be a triple of adapted process such that
\begin{itemize}
  \item [A1] $L, M$ are right-continuous, non-decreasing and $L_0 = M_0 = 0$
  \item [A2] the jumps of $M$ satisfy $\Delta M \leq 1$
  \item [A3] $C \geq 0$ and $\int_0^t C_s ds < \infty$ a.s. for all $t \geq 0$
\end{itemize}
Then, there exists a unique solution $(Z^\nu, Y^\nu, B^\nu)$ to (2.12)-(2.11)-(2.13)-(2.14).
Moreover, $(Z^\nu, Y^\nu, B^\nu)$ satisfies conditions (2.7)-(2.8).

**Proof.** Equation (2.11) clearly defines a unique solution $Y^\nu$. Given $Y^\nu$, it is also clear that (2.13) has a unique solution $Y^\nu B^\nu$, and (2.14) defines $B^\nu$ uniquely. Finally, given $(Y^\nu, B^\nu)$, it is an obvious fact that equation (2.10) has a unique solution $X^\nu$.

**Remark 2.2** The statement of the above proposition is still valid when A2 is replaced by the following weaker condition
\begin{itemize}
  \item [A2'] the jumps of the pair process $(L, M)$ satisfy $\Delta L - \Delta M \geq -1$
\end{itemize}
However in the case where tax credits are allowed by the taxation rule, see Section 5, it is easy to construct consumption-investment strategies, satisfying A1-A2'-A3, which increase without bound the value function of the problem (2.16) defined below, starting from some fixed positive initial holding in stock. Hence such a model allows for a weak notion of arbitrage opportunities. Indeed, for each $\varepsilon > 0$ and $\lambda > 0$, let $\Delta L_t = \Delta M_t := 0$ for $t \neq \tau$, $\Delta L_\tau = \Delta M_\tau := \Lambda$ where $\tau := \inf\{t : B_t > 1 + \varepsilon\}$. By sending $\Lambda$ to infinity, the value function of the problem (2.16) converges to $+\infty$.

**Definition 2.1** Let $\nu = (C, L, M)$ be a triple of $\mathbb{F}$–adapted processes, and $(z, y) \in \tilde{S}$. We say that $\nu$ is a $(z, y)$–admissible consumption-investment strategy if it satisfies Conditions A1-A2-A3 together with the no-bankruptcy condition (2.6). We shall denote by $\mathcal{A}^I(z, y)$ the collection of all $(z, y)$–admissible consumption-investment strategies.

**Remark 2.3** We used the absorption at zero condition in order to express an investment strategy by means of the proportions $dL_t = \frac{d\tilde{L}_t}{Y^\nu_t}$ and $dM_t = \frac{d\tilde{M}_t}{Y^\nu_t}$, instead of the volume of transfers, $d\tilde{L}_t$ and $d\tilde{M}_t$. This modification was needed for the specification of the tax basis.
by means of the process \( K \) defined above. Indeed, in terms of \((\tilde{L}, \tilde{M})\), the dynamics of the state variables \( X \) and \( Y \) are given by:

\[
dY_t = Y_t \frac{dS_t}{S_t} + d\tilde{L}_t - d\tilde{M}_t + d(1 - f(B_{t-})) \tilde{M}_t,
\]

but the relative basis is defined by means of the process \( K \) whose dynamics are given by

\[
dK_t = d\tilde{L}_t - K_t - Y_t - d\tilde{M}_t.
\]

Since the event \( \{Y = 0\} \) has positive probability, this may cause trouble for the definition of the model.

### 2.7 The consumption-investment problem

Throughout this paper, we consider a power utility function:

\[
U(c) := \frac{c^p}{p} \quad \text{for all} \quad c \geq 0,
\]

where \( 0 < p < 1 \) is a given parameter. We next consider the investment-consumption criterion

\[
J^f_t(z, y; \nu) := \mathbb{E} \left[ \int_0^t e^{-\beta t} U(C_s) ds + U(X^f_t) 1_{\{t < \infty\}} \right]
\]

for \( t \in \mathbb{R}_+ \cup \{+\infty\}, (z, y) \in \bar{S} \) and \( \nu \in \mathcal{A}^f(z, y) \). Let

\[
T \in \mathbb{R}_+ \cup \{+\infty\}
\]

be a given time horizon, so that our analysis holds for both finite and infinite horizon. The consumption investment problem is defined by

\[
V^f_T(z, y) := \sup_{\nu \in \mathcal{A}^f(z, y)} J^f_T(z, y; \nu), \quad (z, y) \in \bar{S}.
\]

In the context of financial markets without taxes, i.e. \( f \equiv 0 \), a slight modification of this problem has been solved by Merton [7, 6] by means of a verification argument. In the finite horizon case \( T < \infty \), the tax-free problem can be solved directly by passing to a dual formulation, [8, 1, 5]. In the infinite horizon tax-free problem, Merton [6] singled out the condition

\[
\gamma := \frac{1}{1-p} \left[ \beta - rp - \frac{1}{2} \frac{p}{1-p} \left( \frac{\mu - r}{\sigma} \right)^2 \right] > 0 \quad \text{whenever} \quad T = +\infty,
\]

in order to ensure that the value function is finite. The (explicit) solution in this context is simply obtained by sending the time horizon to infinity in the solution of the finite horizon problem.

We conclude this section by the following easy result which states that, the value function \( V^f \) is non-increasing in \( f \).
Proposition 2.2 Let \((z, y)\) be some initial holdings pair in \(S\), and let \(f \geq g\) be two maps from \(\mathbb{R}_+\) into \(\mathbb{R}\). Then \(V^f_{\mathcal{I}}(z, y)_T \leq V^g_{\mathcal{I}}(z)\).

Proof. Consider some admissible consumption-investment strategy \((C, L, M) \in \mathcal{A}^f(z, y)\). We denote by \((Z^g_t, Y^g_t, B^g_t)\) and \((Z^f_t, Y^f_t, B^f_t)\) the corresponding state process respectively under the taxation rule implied by \(f\) and \(g\). Observing that \((Y^f_t, B^f_t) = (Y^g_t, B^g_t)\), we directly compute that:

\[
Z^g_t - Z^f_t = \int_0^t r(Z^g_s - Z^f_s)ds + \int_0^t [f(B^f_{s-})Y^f_s - g(B^g_{s-})Y^g_s]dM_s \\
\geq \int_0^t r(Z^0_s - Z^f_s)ds + \int_0^t (f - g)(B^f_{s-})Y^f_s dM_s \\
\geq \int_0^t r(Z^0_s - Z^f_s)ds ,
\]

since \(f \geq g\). This shows that \(Z^g \geq Z^f\) so that \((C, L, M)\) is also an admissible strategy in \(\mathcal{A}^g(z, y)\). Hence \(\mathcal{A}^f(z, y) \subset \mathcal{A}^g(z, y)\) by arbitrariness of \((C, L, M) \in \mathcal{A}^f(z, y)\), and the required result follows. \(\square\)

3 Financial market without taxes

In this section, we review the solution of the consumption-investment problem in a financial market without capital gain taxes, i.e. when \(f \equiv 0\). Since the reduction of our problem to this case is slightly different from the classical Merton model, we shall study both problems. We will show that they have essentially the same value functions, and discuss the issue of optimal strategies.

3.1 The classical Merton model

In the classical formulation of the tax-free consumption-investment problem, the investment control variable is described by means of unique process \(\pi\) which represents the proportion of wealth invested in risky assets at each time. Given a consumption plan \(C\), the total wealth process is then defined by the dynamics:

\[
d\bar{Z}^{(C, \pi)}_t = \left(r\bar{Z}^{(C, \pi)}_t - C_t\right)dt + \pi_t \bar{Z}^{(C, \pi)}_t [(\mu - r)dt + \sigma dW_t] , \quad \bar{Z}^{(C, \pi)}_0 = z. \tag{3.1}
\]

In this context, a consumption-investment strategy is a pair of \(\mathbb{F}\)-adapted processes \((C, \pi)\), where \(C\) is non-negative and

\[
\int_0^T C_s ds + \int_0^T |\pi_s|^2 ds < \infty \quad \mathbb{P} - \text{a.s.}
\]

We shall denote by \(\bar{\mathcal{A}}(z)\) the collection of all such consumption-investment strategies which satisfy the additional no-bankruptcy condition

\[
\bar{Z}^{(C, \pi)}_t \geq 0 \quad \mathbb{P} - \text{a.s.} \quad \text{for all} \quad 0 \leq t \leq T .
\]

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The *relaxed* tax-free consumption-investment problem is then defined by:

\[
\bar{V}_T(z) := \sup_{(C, \pi) \in \bar{A}(z)} \mathbb{E} \left[ \int_0^T e^{-\beta t} U(C_t) dt + U\left( \bar{Z}_T^{(C, \pi)} \right) 1_{\{T < \infty\}} \right].
\]

(3.2)

Set \( \partial S := \{(z, y) \in \bar{S} : z = 0\} \), and let \((z, y)\) be an arbitrary initial data in \( S \cup \partial S = \{(z, y) \in \bar{S} : y > 0\} \). Clearly, for any admissible consumption-investment strategy \( \nu = (C, L, M) \in A_0(z, y) \), one can define a pair \((C, \pi) \in \bar{A}(z)\) such that \( Z^\nu = \bar{Z}^{(C, \pi)} \). This shows that

\[
V_0^0(z, y) \leq \bar{V}_T(z) \quad \text{for all} \quad (z, y) \in S \cup \partial S,
\]

(3.3)

and justifies the name of the problem \( \bar{V}_T \). The value function \( V_0^0 \) on the boundary \( \partial S \) will be studied separately.

We shall prove later on (Proposition 3.2) that equality holds in (3.3) by exhibiting a maximizing strategy for the problem \( V_0^0 \). In preparation to this, let us first recall the explicit solution of the relaxed tax-free consumption-investment problem.

**Theorem 3.1** Let Condition (2.17) hold. Then, for all \( z > 0 \):

\[
\bar{V}_T(z) = \frac{z^p}{p} \left[ 1 + \left( 1 - \frac{1}{\gamma} \right) e^{-\gamma t} \right]^{1-p}.
\]

Moreover, existence holds for the problem \( \bar{V}_T(z) \) with optimal consumption-investment strategy given by:

\[
\bar{\pi}_t = \bar{\pi} := \frac{\mu - r}{(1-p)\sigma^2}, \quad \bar{C}_t := \bar{c}(t) \bar{Z}_t,
\]

where \( \bar{c}(.) \) is the deterministic function

\[
\bar{c}(t) := \left[ 1 + \left( 1 - \frac{1}{\gamma} \right) e^{-\gamma (T-t)} \right]^{-1},
\]

and \( \bar{Z} = \bar{Z}^{(C, \bar{\pi})} \) is the wealth process defined by the strategy \((\bar{C}, \bar{\pi})\):

\[
\bar{Z}_0 = z, \quad d\bar{Z}_t = \bar{Z}_t \left[ (r - \bar{c}(t)) dt - \frac{\mu - r}{(1-p)\sigma} \left( \frac{\mu - r}{\sigma} dt + dW_t \right) \right].
\]

Observe that

- the optimal investment strategy \( \bar{\pi} \) is constant both in the finite and infinite horizon cases,
- the optimal consumption process is a linear deterministic function of the wealth process, with slope defined by the function \( \bar{c}(t) \); in the infinite horizon case, the function \( \bar{c} \) reduces to the constant \( \gamma \),

**Remark 3.1** Consider the infinite horizon case \( T = +\infty \). Then \( \bar{c}(t) = \gamma \). By direct computation, we see that

\[
e^{-\beta t} \mathbb{E} \left[ U(\bar{Z}_t) \right] = z^p e^{-\gamma t} \quad \text{for all} \quad t \geq 0.
\]

Hence Condition (2.17) guarantees that \( e^{-\beta t} \mathbb{E} \left[ U(\bar{Z}_t) \right] \to 0 \) as \( t \to \infty \), and therefore:

\[
\bar{V}_\infty(z) = \lim_{t \to \infty} \mathbb{E} \left[ \int_0^t e^{-\beta s} U(\bar{Z}_s) ds + e^{-\beta t} U(\bar{Z}_t) \right].
\]
3.2 Connection with our tax-free model

We now focus on the reduction of the model of Section 2 to the tax-free case, i.e. $f \equiv 0$. In this context, the state variable $B$ is not relevant any more. Given an initial data $(z, y) \in \bar{S}$ and an admissible control $\nu = (C, L, M) \in A^0(z, y)$, the controlled state process reduces to the pair $(Z^\nu, Y^\nu)$ which evolves according to the dynamics:

$$dZ^\nu_t = (rZ^\nu_t - C_t)\,dt + Y^\nu_t\,[(\mu - r)\,dt + \sigma\,dW_t]$$
$$dY^\nu_t = Y^\nu_t\,[(\mu - r)\,dt + \sigma\,dW_t + dL_t - dM_t]$$

together with the initial condition $(Z, Y)_0 = (z, y)$.

This model presents some minor differences with the classical Merton model of Section 3.1. First, the investment strategies are constrained to have bounded variation. We shall see that this induces a non-existence of an optimal control for the problem $V^0(z, y)$, but does not entail any difference between $V^0$ and $\bar{V}$. Second, the above dynamics imply that zero is an absorbing boundary for the $Y$ variable which describes the holdings in stock. From the solution of the classical Merton model reported in Theorem 3.1, notice that the investment in stock is always positive, except the case of zero initial capital $z = 0$. We therefore expect that the value functions $\bar{V}$ and $V^0$ do coincide except on the boundary

$$\partial^0S := \{(z, y) \in \bar{S} : z > 0 \text{ and } y = 0\}.$$ 

Observe that the analysis of both problems $\bar{V}$ and $V^0$ is trivial on the boundary

$$\partial^1S := \{(z, y) \in \bar{S} : z = 0\},$$

since there is no possibility neither for consumption nor for investment. The following result characterizes the value function $V^0$ on the boundary of $S$ which, according to the previous notations, is partitioned into

$$\partial S = \partial^0S \cup \partial^1S.$$

**Proposition 3.1** The solution of the problem $V^0$ on the boundary $\partial S$ is given by:

(i) For $(z, y) \in \partial^0S$,

$$V^0(z, y) = \bar{V}(z) = 0 \quad \text{with optimal controls} \quad (\hat{C}, \hat{L}, \hat{M})_t = (0, 0, 1), \quad t \geq 0,$$

and optimal state process $$(\hat{Z}, \hat{Y})_t = 0 \quad \text{for } t \geq 0.$$ (ii) Set $\gamma_0 := \frac{\beta - rp}{1 - p}$ and assume $\beta > r$ whenever $T = +\infty$. Then, for $(z, y) \in \partial^0S$,

$$V^0_T(z, y) = \frac{z^p}{p} \left[ \frac{1}{\gamma_0} + \left( 1 - \frac{1}{\gamma_0} \right) e^{-\gamma_0 T} \right]^{1-p}, \quad (3.4)$$

with optimal controls

$$(\hat{C}, \hat{L}, \hat{M})_t = (\hat{c}_0(t)\hat{Z}_t, 0, 0), \quad \hat{c}_0(t) := \left[ \frac{1}{\gamma_0} + \left( 1 - \frac{1}{\gamma_0} \right) e^{-\gamma_0 (T - t)} \right]^{-1} \quad (3.5)$$

for $0 \leq t \leq T$; the optimal state processes are $\hat{Y} = 0$ and

$$\hat{Z}_t := z \exp \left[ rt - \int_0^t \hat{c}_0(s)\,ds \right], \quad 0 \leq t \leq T.$$
Proof. For item (i), it is sufficient to observe that the investment strategy \((\hat{C}, \hat{L}, \hat{M})_t = (0, 0, 1), t \geq 0\) is the only admissible strategy. We now concentrate on item (ii). Since \(\partial^0 S\) is an absorbing boundary, we are reduced to the (deterministic) control problem:

\[
V_0^0(t, z) = \sup \int_t^T e^{-\beta t}U(C_t) \, dt + e^{-\beta(T-t)}U(Z_T) \mathbf{1}_{\{T < \infty\}},
\]

where the state dynamics are given by

\[
dZ_t = (rZ_t - C_t)dt.
\]

1. We first solve the finite horizon problem \(T < \infty\). We shall use a verification argument by guessing a solution to the Hamilton-Jacobi equation of this problem:

\[
0 = -\beta V_0^0(t, z) + rZ \frac{\partial V_0^0}{\partial z}(t, z) + \sup_{\xi \geq 0} \left\{ U(\xi) - \xi \frac{\partial V_0^0}{\partial z}(t, z) \right\};
\]

the argument \(y = 0\) has been omitted for notational simplicity. We guess a solution to the above first order partial differential equation in the separable form \(z^p h(t)\), and determine \(h\) so that the terminal condition \(V_0^0(T, z) = U(z)\) or, equivalently, \(h(T) = \frac{1}{p}\) is satisfied. This leads to the candidate solution defined in (3.4). By usual verification arguments, this candidate is then shown to be the solution of the problem, and the optimal controls are identified.

2. We now concentrate on the infinite horizon case \(T = +\infty\). It is clear that the optimal state process \(Z\) should be set to zero at infinity. In terms of optimal control, this is a natural transversality condition for the problem. In order to take advantage of this information, we solve the problem by the calculus of variation approach. Direct calculation from the local Euler equation of the problem leads to the following characterization of the optimal state:

\[
(p - 1) \left( r\tilde{Z} - \dot{Z} \right) = (\beta - r)(rZ - \dot{Z}),
\]

where \(\tilde{Z} = dZ/dt\) denotes the time derivative of the state \(Z\). This ordinary differential equation can be solved explicitly by the technique of variation of the constant. In view of the boundary conditions \(Z_0 = z\) and \(Z_\infty = 0\), this provides the unique solution to the local Euler equation:

\[
Z_t = z \exp \left( \frac{\beta - r}{1 - p} \right) t \quad \text{with optimal consumption} \quad C_t = rZ_t - \dot{Z}_t = \gamma_0 Z_t.
\]

Notice that the condition \(\beta > r\) is here necessary in order to ensure that \(Z_\infty = 0\).

To conclude our analysis of the tax-free model, we now focus on the value function in the interior of the domain \(S\). In contrast with the situation on the boundary, trading in the
stock is now possible. The following result shows that the value function \( V^0 \) coincides with \( \bar{V} \), the maximal utility in the classical Merton model. The price for the control restriction to the class of bounded variation processes is that existence does not hold any more for the problem \( V^0 \).

**Proposition 3.2** For all \((z, y) \in S\), we have \( V^0_T(z, y) = \bar{V}_T(z) \).

In view of (3.3), the only non-trivial inequality in the above result is that \( V^0_T(z, y) \geq \bar{V}_T(z) \). This follows directly from our main Theorem 4.1, by considering the case \( f \equiv 0 \).

### 4 Optimal consumption-investment under capital gain taxes

In this section, we consider the case of a financial market with no tax credits, i.e.

\[
f(b) \geq 0 \quad \text{for all} \quad b \geq 0.
\]  

(4.1)

The general case will be studied in the subsequent section.

In the context of (4.1), we shall prove the first main result of the paper which states that the maximal utility in the financial market is not altered by the capital gains taxation rule, i.e. \( V^f = V^0 \).

This result is of course trivial when the initial holding in stock is zero, \( Y_0 = 0 \), since \( \partial S \) is an absorbing boundary. For a non-zero initial holding \( y \) in stock, we shall prove this result by forcing the relative tax basis \( B_t \) to be as close as desired to unity, and tracking Merton’s optimal strategy, i.e. keep the proportion of wealth invested in the risky asset

\[
\pi_t := \frac{Y_t}{Z_t} \mathbf{1}_{\{Z_t \neq 0\}}, \quad 0 \leq t \leq T,
\]

and the proportion of wealth dedicated for consumption

\[
c_t := \frac{C_t}{Z_t} \mathbf{1}_{\{Z_t \neq 0\}}, \quad 0 \leq t \leq T,
\]

close to the pair \((\bar{\pi}, \bar{c}(t))\) defined in Theorem 3.1.

To do this, we first fix some \( t > 0 \), and define a convenient sequence \((\nu^{t,n})_{n \geq 1} := (C^{t,n}, L^{t,n}, M^{t,n})_{n \geq 1}\) for all \((z, y) \in S \cup \partial^2 S\). We shall denote by \((Z^{t,n}, Y^{t,n}, B^{t,n}) = (Z^{t,n}, Y^{t,n}, B^{t,n})\) the corresponding state processes. For each integer \( n \geq 1 \), the consumption-investment strategy \( \nu^{t,n} \) is defined as follows.

1. At time 0 choose the transfers \((\Delta L_0^{t,n}, \Delta M_0^{t,n})\) so as to adjust the proportion of wealth to \( \bar{\pi} \):

\[
\Delta L_0^{t,n} := \left(\bar{\pi} \frac{z}{y} - 1\right) \mathbf{1}_{\{z \geq y\}} \quad \text{and} \quad \Delta M_0^{t,n} := \left(1 - \frac{z}{y}\right) \mathbf{1}_{\{z < y\}},
\]

so that

\[
\bar{\nu}_0^{t,n} := \frac{Y_0^{t,n}}{Z_0^{t,n}} = \bar{\pi}.
\]
recall that $B_{0-} = B_{0-} = 1$.

2. At the final time $t$, fix the jumps $(\Delta L^{t,n}, \Delta M^{(t,n)})$ so that all the wealth is transferred to the bank:

$$\Delta L^{t,n}_t := 0 \quad \text{and} \quad \Delta M^{(t,n)}_t = 1.$$  

This implies that

$$Y^{t,n}_t = 0 \quad \text{and} \quad Z^{t,n}_t = X^{(t,n)}_t.$$  

3. In Step 3 below, we shall construct a sequence of stopping times $(\tau^{t,n}_k)_{k \geq 1}$. Our consumption strategy is defined by

$$C^{t,n}_s := \bar{c}(t)Z^{t,n}_s \quad \text{for} \quad 0 \leq s \leq T.$$  

The investments strategy is piecewise constant:

$$dL^{t,n}_s = dM^{t,n}_s = 0 \quad \text{for all} \quad s \in [0, T] \setminus \{\tau^{t,n}_k, k \geq 1\}.$$  

4. We now introduce the sequence of stopping times $\tau^{t,n}_k$ as the hitting times of the pair process $(\pi, B)$ of some barrier close to $(\bar{\pi}, 1)$. Set $\tau^{t,n}_0 := 0$, and define the sequence of stopping times

$$\tau^{t,n}_k := T \wedge \tau^\pi_k \wedge \tau^B_k,$$

where

$$\tau^\pi_k := \inf \left\{ s \geq \tau^{t,n}_{k-1} : |\pi^{t,n}_s - \bar{\pi}| > n^{-1} \right\},$$

$$\tau^B_k := \inf \left\{ s \geq \tau_{k-1} : (1 - B^{t,n}_s) > n^{-1} \lambda^k \right\},$$

where $\lambda$ is a parameter in $(0, 1)$ to be fixed later on.

5. To conclude the definition of $\nu^{t,n}$, it remains to specify the jumps $(\Delta L^{t,n}, \Delta M^{t,n})$ at each time $\tau^{t,n}_k$. The idea here is to re-set the proportion $\pi^{t,n}$ to the constant $\bar{\pi}$, and to push-back the relative tax basis $B$ to unity. To do this, we first consider some parameter $\lambda \in (0, 1)$ such that:

$$1 + \lambda (1 - \bar{\pi}) > 0.$$  

We then define for all $s \in \{\tau^{t,n}_k, k \geq 1\}$:

$$\Delta L^{t,n}_s := \frac{\bar{\pi} - \pi^{t,n}_s - f\left( B^{t,n}_{s-} \right)}{\pi^{t,n}_s - 1 + \lambda \left[ 1 - \bar{\pi} f\left( B^{t,n}_{s-} \right) \right]} \quad \text{and} \quad \Delta M^{t,n}_s := 1 - \lambda \Delta L^{t,n}_s.$$  

Using the dynamics of $(Z, Y, B)$, we have:

$$\pi^{t,n}_s = \frac{Y^{t,n}_s}{Z^{t,n}_s} = \frac{\frac{f\left( B^{t,n}_{s-} \right) - \Delta L^{t,n}_s}{1 - \bar{\pi} f\left( B^{t,n}_{s-} \right)}}{\pi^{t,n}_{s-} - \Delta M^{t,n}_s}.$$
Let \( \tau \) be some fixed time horizon. Then for any map \( f \) satisfying \((2.5)\), we have
\[
B^{t,n}_s Y^{t,n}_s - B^{t,n}_{s-} Y^{t,n}_{s-} = Y^{t,n}_s \left( \Delta L^{t,n}_s - B^{t,n}_{s-} \Delta M^{t,n}_s \right),
\]
so that with the above definition of the jumps \((\Delta L^{t,n}, \Delta M^{t,n})\), we have
\[
\pi^{t,n}_s = \bar{\pi} \text{ and } B^{t,n}_s = \frac{1 + \lambda B^{t,n}_{s-}}{1 + \lambda} \text{ for } s \in \{\tau^{t,n}_k, \ k \geq 0\}.
\]

**Remark 4.1** Since \( f(1) = 0 \) and \( f \) is continuous, it is immediately checked from the above definitions that, for sufficiently large \( n \):
\[
0 < \Delta L^{t,n}_s < 1 \text{ and } 0 < \Delta M^{t,n}_s < 1 \text{ for } s \in \{\tau^{t,n}_k, \ k \geq 0\}.
\]
This guarantees that the process of holdings in risky assets \( Y^{t,n} \) is positive \( \mathbb{P} \)-a.s. 

**Remark 4.2** The sequence \( (\tau^{t,n}_k)_{k \geq 0} \) is strictly increasing, and converges to \( T \). To see this, we first make the trivial observation that \( \tau^{t,n}_k < \tau^{t,n}_{k+1} \) \( \mathbb{P} \)-a.s. On the other hand, since \( L \) and \( M \) are constant in the stochastic interval \([\tau^{t,n}_k, \tau^{t,n}_{k+1})\), we have \( 1 - B^{t,n}_{\tau^{t,n}_k} \leq \lambda^k/n \). Then :
\[
1 - B^{t,n}_{\tau^{t,n}_k} = \frac{\lambda}{1 + \lambda} \left( 1 - B^{t,n}_{\tau^{t,n}_{k-}} \right) \leq \frac{\lambda^{k+1}}{n(1 + \lambda)} < \frac{\lambda^{k+1}}{n}.
\]
This guarantees that \( \tau^{t,n}_k < \tau^{t,n}_{k+1} \mathbb{P} \)-a.s. In particular \( \tau^{t,n}_k \to \tau^{t,n}_\infty \leq T \). The proof of our claim is completed by observing that the limit \( \tau^{t,n}_\infty \) is necessarily equal to \( T \). 

The main property of the sequence \( (u^{t,n})_n \) is the following.

**Lemma 4.1** Let \( t > 0 \) be some fixed time horizon. Then for any map \( f \) satisfying \((2.5)\), we have
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| Z^{t,n}_s - \bar{Z}_s \right|^2 \right] \leq n^{-2} \alpha e^{\alpha t},
\]
for some constant \( \alpha \) depending on \( t \).

**Proof.** By definition of the sequence of stopping times \( \left(\tau^{t,n}_k\right)_k \), we have
\[
\sup_{0 \leq s \leq t} \left| \pi^{t,n}_s - \bar{\pi} \right| \leq \frac{1}{n} \text{ and } \sup_{\tau^{t,n}_{k-1} \leq s < \tau^{t,n}_k} \left| 1 - B^{t,n}_s \right| \leq \frac{\lambda^k}{n} \text{ for all } k \geq 1.
\]

Set \( D := Z^{t,n} - \bar{Z} \). Since \( D_0 = 0 \), we decompose \( D \) into :
\[
D_s = F_s + G_s + H_s,
\]
where
\[
F_s := \int_0^s D_u \left[ (r - c(u))du + \pi^{t,n}_u ((\mu - r)du + \sigma dW_u) \right],
\]
\[
G_s := \int_0^s \bar{Z}_u \left( \pi^{t,n}_u - \bar{\pi} \right) ((\mu - r)du + \sigma dW_u)
\]
\[
H_s := - \int_0^s \pi^{t,n}_u f \left( B^{t,n}_{u-} \right) \left( D_u - \bar{Z}_{u-} \right) dM^{t,n}_u = \sum_{u \leq s} \pi^{t,n}_u \cdot f \left( B^{t,n}_{u-} \right) \left( D_u - \bar{Z}_{u-} \right) \Delta M^{t,n}_u.
\]
In the subsequent calculation, \( A \) will denote a generic \((t-\)dependent) constant whose value may change from line to line. We shall also denote by \( V^* := \sup_{0 \leq u \leq s} |V_u| \) for all process \((V_s)_s\).

We first start by estimating the first component \( F \). Observe that \( \bar{c}(.) \) is bounded and the process \( \pi^{t,n} \) is bounded by \( 2 \bar{\pi} \) for large \( n \). Then

\[
|F_s|^2 \leq 2 \left( \int_0^s D_u (r - \bar{c}(u) + \pi^{t,n}_u (\mu - r)) du \right)^2 + 2 \left( \int_0^s D_u \pi^{t,n}_u \sigma dW_u \right)^2
\]

By the Buckholder-Davis-Gundy inequality, this provides

\[
E|F_s^*|^2 \leq A \left( \int_0^s E|D_u^*|^2 du + E \int_0^s |D_u|^2 |\pi^{t,n}_u|^2 \sigma^2 du \right) \leq A \int_0^s E|D_u^*|^2 du . \quad (4.3)
\]

Similarly, it follows from (4.2) that:

\[
|G_s|^2 \leq \frac{2}{n^2} \left( (\mu - r)^2 \left( \int_0^s \bar{Z}_u du \right)^2 + \sigma^2 \left( \int_0^s \bar{Z}_u dW_u \right)^2 \right) .
\]

Using again the Buckholder-Davis-Gundy inequality, this provides

\[
E|G_s^*|^2 \leq \frac{A}{n^2} . \quad (4.4)
\]

Finally, since the jumps of \( M^{t,n} \) are bounded by 1, we estimate the component \( H \) by:

\[
|H_s|^2 \leq 2 \left( \sum_{u \leq s} \pi^{t,n}_u |f \left( B^{t,n}_{u,s-} \right) |D_u-| \right)^2 + 2 \left( \sum_{u \leq s} f \left( B^{t,n}_{u,s-} \right) \bar{Z}_u- \right)^2
\]

where the last inequality follows from (4.2) and (2.5) which implies that \( f \) is locally Lipschitz at \( b = 1 \). We now collect the estimates from (4.3), (4.4) and (4.5) to see that:

\[
\left( 1 - \frac{A}{n^2} \right) E|D_s^*|^2 \leq \frac{A}{n^2} + K \int_0^s \mathbb{E}|D_u^*|^2 du \quad \text{for all} \quad s \leq t .
\]
The required result follows from the Gronwall inequality.

We are now ready for the first main result of this paper which states that the value function of the consumption investment problem is not altered by the capital gain taxes rule. Notice that the proof produces a precise description of optimal consumption-investment behavior: in the finite horizon case \( T < \infty \), \((\nu^{T,n})_n\) is a maximizing consumption-investment strategy, in the infinite horizon case, a maximizing consumption-investment strategy is obtained by means of a diagonal extraction argument from the sequence \((\nu^{t,n})_n\).

**Theorem 4.1** Let \( T \in \mathbb{R}_+ \cup \{+\infty\} \) be some given maturity. Assume the the function \( f \) defining the taxation rule satisfies (2.5) and (4.1). Then \( V^f_T = V^0_T \) on \( \bar{S} \). In particular, in \( S \cup \partial S \), the value function of the consumption investment problem under taxes coincides with the value function of the classical Merton problem.

**Proof.** 1. We first show that

\[
\lim_{n \to \infty} J^f_t(z, y; \nu^{t,n}) = \bar{J}_t(z) := \mathbb{E} \int_0^t e^{-\beta s} U(C_s \bar{Z}_s) ds + e^{-\beta T} U(\bar{Z}_t)
\]

for all \((z, y)\) be in \( S \) and \( t \in \mathbb{R}_+ \). Indeed, since the utility function \( U \) is \( p\)-holder continuous

\[
|U(\bar{c}(s)Z_s^{t,n} - U(\bar{c}(s)\bar{Z}_s)| \leq \bar{c}(s)|Z_s^{t,n} - \bar{Z}_s|^p
\]

Then using the Jensen inequality with the concave function \( x \mapsto x \frac{p}{2} \):

\[
\mathbb{E}|Z_s^{t,n} - \bar{Z}_s|^p = \mathbb{E}(|Z_s^{t,n} - \bar{Z}_s|^2)^{\frac{p}{2}} \leq (\mathbb{E}|Z_s^{t,n} - \bar{Z}_s|^2)^{\frac{p}{2}}
\]

Now, using the estimate provided by lemma (4.1)

\[
(\mathbb{E}|Z_s^{t,n} - \bar{Z}_s|^2)^{\frac{p}{2}} \leq (n^{-2} \alpha e^{\alpha t})^{\frac{p}{2}} = n^{-p} \alpha e^{\alpha t}
\]

It follows that:

\[
|\bar{J}_t(z) - J^f_t(z, y; \nu^{t,n})| \leq n^{-p} \left( \frac{2}{p} c(0)^p + \alpha \right) e^{\alpha t} + \frac{1}{p} \int_0^t e^{\frac{p}{2} \alpha s} ds.
\]

2. Combining Proposition 2.2 together with (3.3), we see that \( V^f \leq V^0 \leq \bar{V} \) on \( S \cup \partial S \). In order to prove that equality holds, it suffices to show that \( V^f \geq \bar{V} \). In the finite horizon case, the proof is completed by taking \( t = T \) in Step 1. We next concentrate on the infinite horizon case \( T = +\infty \). Fix some positive integer \( k \). By Remark 3.1, we have:

\[
\bar{V}_T(z) = \lim_{t \to \infty} \bar{J}_t(z).
\]

Then

\[
\bar{J}_{t_k}(z) \geq \bar{V}_T(z) - \frac{1}{k},
\]

for some \( t_k > 0 \). By the first step of this proof:

\[
\lim_{n \to \infty} J^f_{t_k}(z, y; \nu^{t_k,n}) = \bar{J}_{t_k}(z).
\]
Then, there exists some integer $n_k$

$$J^I_{tk}(z, y; \nu^{\ell_k,n_k}) \geq \bar{J}_{tk}(z) - \frac{1}{2k} \geq \bar{V}_T(z) - \frac{1}{k}.$$ 

3. Finally, we define the consumption-investment strategies $\hat{\nu}^k$ consisting in following $\nu^{\ell_k,n_k}$ up to $t_k$, then liquidating at $t_k$ the risky asset position and making a null consumption on the time interval $(t_k, T)$. Then:

$$J^I_{T}(z, y; \hat{\nu}^k) \geq J^I_{tk}(z, y; \nu^{\ell_k,n_k}) \geq \bar{V}_T(z) - \frac{1}{k}$$ for all $k \geq 0$.

This proves that $V^0_{T}(z, y) \geq \limsup_{k \to \infty} J^I_{T}(z, y; \hat{\nu}^k) \geq \bar{V}_T(z)$.

\[ \Box \]

5 Extension to Taxation rules with possible tax credits

In this section we consider the case where the financial market allows for tax credits, and we restrict our analysis to the case

$$\delta = 0.$$

Our main purpose is to extend Theorem 4.1 to this context.

**Theorem 5.1** Consider a taxation rule defined by the map $f$ satisfying (2.5), and let $\delta = 0$. Assume further that

$$\inf_{b \geq 0} \frac{f(b)}{1-b} > -\infty . \tag{5.1}$$

Then, for all $T \in \mathbb{R} \cup \{+\infty\}$ and $(z, y) \in \bar{S}$, we have $V^{f}_{T}(z, y) = V^{0}_{T}(z, y)$.

**Proof.** 1. Since $f \leq f^+$, $V^{f}_{T} \geq V^{f^+}_{T}$ by Proposition 2.2. Now $f^+$ defines a taxation rule without tax credits as required by Condition (4.1). It then follows from Theorem 4.1 that $V^{f}_{T} \geq V^{0}_{T}$.

2. From (5.1) it follows that $V^{f}_{T} \leq V^{\ell}_{T}$, where $\ell$ is the linear map

$$\ell(b) := \alpha (1-b) \quad \text{with} \quad \alpha := \inf_{b \geq 0} \frac{f(b)}{1-b} .$$

We shall prove in Proposition 5.2 that $V^{f}_{T} = V^{0}_{T}$, hence $V^{f}_{T} \leq V^{0}_{T}$.

\[ \Box \]

In the above proof, we reduced the problem to the linear taxation rule of Example 2.3. The rest of this section is specialized to this context.

5.1 Immediate realization of capital losses for the linear taxation rule

Consider the taxation rule defined by

$$f(b) := \alpha (1-b) \quad \text{for all} \quad b \geq 0 . \tag{5.2}$$
We first intend to prove that it is always worth realizing capital losses whenever the tax basis exceeds unity. In other words, for each \((z, y)\) in \(\overline{S}\), any admissible consumption-investment strategy for which the relative tax basis exceeds 1 at some stopping time \(\tau\) can be improved strictly by increasing the sales of the risky asset at \(\tau\). We shall refer to this property as the \textit{optimality of the immediate realization of capital losses}. In a discrete-time framework, this property was stated without proof by [4].

**Proposition 5.1** Let \(f\) be the linear taxation rule of (5.2), \(\delta = 0\), and consider some initial holdings \((z, y)\) in \(\overline{S}\). Consider some consumption-investment strategy \(\nu := (C, L, M)\) in \(\mathcal{A}^f(z, y)\), and suppose that there is a finite stopping time \(\tau \leq T\) with \(P[\tau < T] > 0\) and \(B_\nu^\tau > 1\) a.s. on \(\{\tau < T\}\).

Then there exists an admissible strategy \(\bar{\nu} = \mathcal{V}(\nu, \tau)\) such that

\[
Y^\bar{\nu} = Y^\nu, \quad Z^\bar{\nu} \geq Z^\nu, \quad B^\bar{\nu} \leq B^\nu, \quad \bar{C} \geq C,
\]

and

\[
J^f_T(z, y; \bar{\nu}) > J^f_T(z, y; \nu).
\]

We start by proving the following lemma which shows how to take advantage of the tax credit at time \(\tau\).

**Lemma 5.1** Let \(f\) be as in (5.2), \(\delta = 0\), and consider some initial holdings \((z, y)\) be in \(\overline{S}\). Consider some consumption-investment strategy \(\nu := (C, L, M)\) in \(\mathcal{A}^f(z, y)\), and suppose that there is a finite stopping time \(\tau \leq T\) with \(P[\tau < T] > 0\) and \(B_\nu^\tau > 1\) a.s. on \(\{\tau < T\}\).

Define \(\bar{\nu} = (\bar{C}, \bar{L}, \bar{M})\) by

\[
\bar{C} := C \quad \text{and} \quad (\bar{L}, \bar{M}) := (L, M) + (1, 1)(1 - \Delta M_\tau) \mathbf{1}_{t \geq \tau}.
\]  

Then \(\bar{\nu} \in \mathcal{A}^f_T(z, y)\) and the resulting state processes are such that

\[
Y^\bar{\nu} = Y^\nu, \quad Z^\bar{\nu} \geq Z^\nu, \quad B^\bar{\nu} \leq B^\nu,
\]

and, almost surely,

\[
B^\bar{\nu}_\tau = 1, \quad Z^\bar{\nu}_\tau > Z^\nu_\tau \quad \text{on} \quad \{\tau < T\}.
\]

**Proof.** 1. Since \(\nu\) and \(\bar{\nu}\) differ only by the jump at the stopping time \(\tau\), and \(\Delta \bar{L}_\tau = \Delta \bar{M}_\tau\), we have

\[
Y^\bar{\nu} = Y^\nu,
\]

and

\[
(Z^\bar{\nu}_t, B^\bar{\nu}_t) = (Z^\nu_t, B^\nu_t) \quad \text{for all} \quad t < \tau.
\]

By the definition of \(\bar{\nu}\), it is easily seen that

\[
B^\bar{\nu}_\tau = 1.
\]
and
\[ Z^\nu_t - Z^\nu_r = -f \left( B^\nu_r \right) Y^\nu_r (1 - \Delta M_r) = \alpha Y^\nu_r \left( B^\nu_r - 1 \right) (1 - \Delta M_r). \] (5.4)

Since
\[ 0 > (1 - B^\nu_r) = (1 - B^\nu_r) \frac{1 - \Delta M_r}{1 + \Delta L_r - \Delta M_r} \text{ on } \{ \tau < T \}, \] (5.5)
it follows that \( B_\tau > 1 \) and \( 1 - \Delta M_r > 0 \) a.s. on \( \{ \tau < T \} \). We therefore deduce from (5.4) that
\[ Z^\nu_\tau - Z^\nu_r > 0 \text{ on } \{ \tau < T \}. \]

2. We next examine the state variable \( B^\nu_t \) for \( t > \tau \). Since \( Y^\nu = Y^\nu \), we have \( K^\nu - K^\nu = Y^\nu (B^\nu - B^\nu) \), and
\[ d(K^\nu_t - K^\nu_t) = -(K^\nu_t - K^\nu_t) \, dM_t, \quad \text{with } K^\nu_t - K^\nu_t = Y^\nu (1 - B^\nu). \] (5.6)

This linear stochastic differential equation can be solve explicitly:
\[ K^\nu_t - K^\nu_t = (K^\nu_t - K^\nu_t) e^{-M^\nu_t + M^\nu_t} \prod_{\tau \leq u \leq t} (1 - \Delta M_u), \] for all \( t \geq \tau \),
where \( M^\nu \) denotes the continuous part of \( M \). Since \( K^\nu_t - K^\nu_t = Y^\nu (1 - B^\nu) < 0 \) on \( \{ \tau < T \} \), this shows that
\[ K^\nu_t \leq K^\nu_t \text{ and therefore } B^\nu_t \leq B^\nu_t \text{ for } t \geq \tau. \] (5.7)

3. In this step, we intend to prove that \( Z^\nu_t \geq Z^\nu_t \) for \( t \geq \tau \). From the dynamics of the processes \( Z^\nu \) and \( Z^\nu \) we have, for \( t > \tau \):
\[ e^{-r(t-\tau)} (Z^\nu_t - Z^\nu_r) = Z^\nu_t - Z^\nu_r - \int_\tau^t e^{-r(u-\tau)} Y^\nu_u \left( f(B^\nu_u) - f(B^\nu_u) \right) \, dM_u \]
\[ = Z^\nu_t - Z^\nu_r + \alpha \int_\tau^t e^{-r(u-\tau)} Y^\nu_u \left( B^\nu_u - B^\nu_u \right) \, dM_u \]
\[ = Z^\nu_t - Z^\nu_r + \alpha \int_\tau^t e^{-r(u-\tau)} \left( K^\nu_u - K^\nu_u \right) \, dM_u, \]
where the last equality follows from the fact that \( Y^\nu = Y^\nu \). We next use (5.7) and (5.6) to see that, for \( t \geq \tau \):
\[ e^{-r(t-\tau)} (Z^\nu_t - Z^\nu_r) \geq Z^\nu_t - Z^\nu_r + \alpha \int_\tau^t \left( K^\nu_u - K^\nu_u \right) \, dM_u \]
\[ = Z^\nu_t - Z^\nu_r + \alpha \left( K^\nu_r - K^\nu_r - K^\nu_t + K^\nu_t \right). \]

Now, since \( Y^\nu_r = Y^\nu_r \) \( (1 + \Delta L_r - \Delta M_r) \), it follows from (5.4) and (5.5) that \( Z^\nu_\tau - Z^\nu_\tau = -\alpha \left( K^\nu_\tau - K^\nu_\tau \right) \). Hence:
\[ e^{-r(t-\tau)} (Z^\nu_t - Z^\nu_r) \geq -\alpha \left( K^\nu_t - K^\nu_t \right) \geq 0 \text{ for } t \geq \tau, \]
by (5.7).

4. Clearly, \( \tilde{\nu} \) satisfies Conditions A1-A2-A3, and \( Z^0 \geq Z^\nu \) by the previous steps of this proof. Hence \( \tilde{\nu} \in \mathcal{A}^f(z, y) \).

**Proof of Proposition 5.1.** Consider the slight modification \( \tilde{\nu} = (\tilde{C}, \tilde{L}, \tilde{M}) \) of the consumption-investment strategy introduced in Lemma 5.1:

\[
\tilde{C}_t := \bar{C}_t + \xi \left( Z^\nu_t - Z^0_t \right) 1_{t \geq \tau} \quad \text{and} \quad \left( \tilde{L}, \tilde{M} \right) := (\bar{L}, \bar{M}), \tag{5.8}
\]

where \( \xi \) is an arbitrary positive constant. Observe that \( (Y^\tilde{\nu}, B^\tilde{\nu}) = (Y^\bar{\nu}, B^\bar{\nu}) \), and \( Z^\tilde{\nu}_t = Z^\bar{\nu}_t \) for \( t \leq \tau \). In order to check the admissibility of the triple \( (\tilde{C}, \tilde{L}, \tilde{M}) \), we repeat Step 3 of the proof of Lemma 5.1:

\[
e^{-r(t-\tau)} \left( Z^\tilde{\nu}_t - Z^\nu_t \right) = Z^\tilde{\nu}_t - Z^\nu_t - \int_\tau^t e^{-r(u-\tau)} Y^\nu_{u-} \left( f(B^\nu_{u-}) - f(B^\nu_{u-}) \right) dM_u + \xi \int_\tau^t e^{-r(u-\tau)} \left( Z^\tilde{\nu}_u - Z^\nu_u \right) du \]
\[
= Z^\tilde{\nu}_t - Z^\nu_t - \int_\tau^t e^{-r(u-\tau)} Y^\nu_{u-} \left( f(B^\nu_{u-}) - f(B^\nu_{u-}) \right) dM_u + \xi \int_\tau^t e^{-r(u-\tau)} \left( Z^\tilde{\nu}_u - Z^\nu_u \right) du \]

\[
\cdots \geq \xi \int_\tau^t e^{-r(u-\tau)} \left( Z^\tilde{\nu}_u - Z^\nu_u \right) du.
\]

Since \( Z^\tilde{\nu}_t - Z^\nu_t = Z^\tilde{\nu}_t - Z^\nu_t \geq 0 \) by Lemma 5.1, it follows from the Gronwall lemma that \( Z^\tilde{\nu}_t \geq Z^\nu_t \) a.s. Hence \( \tilde{\nu} \in \mathcal{A}^f_T(z, y) \).

Recall from Lemma 5.1 that \( Z^\tilde{\nu}_\tau > Z^\nu_\tau \) on \( \{ \tau < T \} \). Since the process \( (Z^\tilde{\nu}_t - Z^\nu_t) \) is right-continuous, the strict inequality holds on some nontrivial time interval almost surely on \( \{ \tau < T \} \). Hence \( \tilde{C} > C \) with positive Lebesgue\( \otimes \)\( P \) measure, and

\[
\mathcal{J}^f_T(z, y; \tilde{\nu}) > \mathcal{J}^f_T(z, y; \nu).
\]

\[\square\]

### 5.2 Reduction to the tax-free financial market

In view of the optimality of the immediate realization of capital losses stated in Proposition 5.1, we expect that, given \( (z, y) \in \mathcal{S} \), and for all constant \( \varepsilon > 0 \), the problem of maximizing \( \mathcal{J}^f_T(z, y; \nu) \) can be restricted to those admissible control processes \( \nu \) inducing a cumulated tax credit bounded by \( \varepsilon \). This result, stated in Lemma 5.2, will allow to prove that \( V^f = V^0 \) in the context of a linear taxation rule, hence completing the proof of Theorem 5.1.

**Lemma 5.2** Let \( f \) be the linear taxation rule defined in (5.2), \( \delta = 0 \), and let \( t > 0 \) be some finite maturity, \( \varepsilon > 0 \), and \( \nu \) in \( \mathcal{A}^f(z, y) \). Then, there exists \( \nu^\varepsilon = (C^\varepsilon, L^\varepsilon, M^\varepsilon) \) in \( \mathcal{A}^f(z, y) \) such that

\[
\mathcal{J}^f_T(z, y; \nu^\varepsilon) \geq \mathcal{J}^f_T(z, y; \nu) \quad \text{and} \quad \int_0^t (B^\nu_{u-} - 1) Y^\nu_{u-} dM^\nu_{u-} \leq \varepsilon \quad \text{a.s.}
\]

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Proof. Let $\theta^0 := 0$, $\nu^0 := \nu$, 
\[
\theta^{n+1} := t \wedge \inf\{s > \theta^n : (B_{s}^{\nu^n} - 1) (1 \vee H_{s}^{\nu^n}) > \varepsilon\},
\]
with 
\[
H_{s}^{\nu^n} := Y_{s}^{\nu^n} e^{M_{s}^{\nu^n} - c} \prod_{u \leq s} (1 - \Delta M_{u}^{\nu^n})^{-1},
\]
and 
\[
\nu^{n+1} := \nu^n 1_{\{\theta^{n+1} = t\}} + \mathcal{V}(\nu^n, \theta^{n+1}) 1_{\{\theta^{n+1} < t\}}.
\]
where $M_{s}^{\nu^n, c}$ denotes the continuous part of $M_{s}^{\nu^n}$, and $\mathcal{V}$ is defined in Proposition 5.1 so as to take advantage of the tax credit while decreasing the relative tax basis. We shall simply denote $(Z^n, Y^n, B^n) := (Z^{\nu^n}, Y^{\nu^n}, B^{\nu^n})$. Then:
\[
C^{n+1} \geq C^n, \ Y^n = Y^0, \ Z^{n+1} \geq Z^n, \ B^{n+1} \leq B^n,
\]
and
\[
(B_{s}^{\nu^n} - 1) (1 \vee H_{s}^{\nu^n}) \leq \varepsilon \ \text{for} \ t \leq \theta^{n+1}.
\]
Clearly, for a.e. $\omega \in \Omega$, $\theta^n(\omega) = t$ for $n \geq N(\omega)$, where $N(\omega)$ is some sufficiently large integer. Therefore the sequence $\nu^n(\omega)$ is constant for $n \geq N(\omega)$ and 
\[
\nu^n \longrightarrow \nu^\varepsilon \ \text{a.s.}
\]
for some $\nu^\varepsilon \in \mathcal{A}^f(z, y)$. Also, by construction of the sequence $\nu^n$, we have:
\[
\nu_s^\varepsilon = \nu_s^n \ \text{for} \ s \leq \theta^{n+1}.
\]
By (5.9), it is immediately checked that $\mathcal{J}^f(z, y; \nu^\varepsilon) \geq \mathcal{J}^f(z, y; \nu)$. We finally use (5.10) and (5.11), together with Itô’s lemma, to compute that:
\[
\int_0^t (B_u^{\nu^\varepsilon} - 1) Y_u^{\nu^\varepsilon} dM_u^{\nu^\varepsilon} \leq \varepsilon \int_0^t Y_{u-}^\nu (H_u^\varepsilon)^{-1} dM_u^{\nu^\varepsilon} = \varepsilon \int_0^t e^{-M_u^{\nu^\varepsilon}} \prod_{u \leq s} (1 - \Delta M_u^{\nu^\varepsilon}) dM_u^{\nu^\varepsilon} = \varepsilon \left[ 1 - e^{-M_t^{\nu^\varepsilon}} \prod_{u \leq t} (1 - \Delta M_u^{\nu^\varepsilon}) \right] \leq \varepsilon,
\]
by the fact that $1 - \Delta M_u^{\nu^\varepsilon} \leq 1$. \hfill $\square$

We are now ready to state the extension of Theorem 4.1 to the linear taxation rule case with tax credits.
Proposition 5.2. Consider the linear taxation rule of (5.2), and let \( \delta = 0 \). Then, for all \( T \in \mathbb{R} \cup \{+\infty\} \) and \((z, y) \in \mathcal{S}\), we have \( V^f_T(z, y) = V^0_T(z, y) \).

Proof. The result is trivial for \((z, y) \in \partial \mathcal{S}\). We then assume in the sequel that \((z, y) \in \mathcal{S} \cup \partial \mathcal{S}\).

1. Since \( f \leq f^+: = \max\{f, 0\} \), we have \( V^f_T \geq V^{f^+}_T \). Now \( f^+ \) defines a taxation rule without tax credits as required by Condition (4.1). It then follows from Theorem 4.1 that \( V^{f^+}_T = V^0_T \) and therefore \( V^f_T \geq V^0_T \).

2. We next concentrate on the reverse inequality. Notice that

\[
\mathcal{J}^f_T(z, y, \nu) \leq \liminf_{t \to -\infty} \mathcal{J}^f_t(z, y, \nu).
\]

This follows by the monotone convergence theorem and the fact that the utility function is non-negative. Therefore, in order to prove the required inequality, it is sufficient to show that, for any fixed finite maturity \( 0 < t \leq T \):

\[
\mathcal{J}^f_t(z, y; \nu) \leq V^0_T(z, y) \quad \text{for all} \quad \nu \in \mathcal{A}^f(z, y) .
\]  

(5.12)

Let \( \varepsilon > 0 \), and consider the consumption-investment strategy \( \nu^\varepsilon \in \mathcal{A}^f(z, y) \) defined in Lemma 5.2. Let \( Z^\varepsilon \) be the wealth process induced by the strategy \( \nu^\varepsilon \) in a tax-free market, i.e. with \( f \equiv 0 \). Since the taxation rule \( f \) allows for tax credits, there is no reason for the process \( Z^\varepsilon \) to be non-negative. Using the dynamics of \( Z^\nu \), it follows from Lemma 5.2 that

\[
e^{-ru}(Z^\varepsilon_u - Z^\nu_u) = \alpha \int_0^u (1 - B_s^{\nu^\varepsilon}) Y_s^{\nu^\varepsilon} dM_s^{\nu^\varepsilon} \geq -\alpha \varepsilon .
\]

(5.13)

Now, from Lemma 5.2 and the increase of the utility function \( U \) and its concavity, we have

\[
\mathcal{J}^f_t(z, y; \nu) \leq \mathcal{J}^f_t(z, y; \nu^\varepsilon) \leq \mathcal{J}^f_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) = \mathcal{J}^0_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) + e^{-\beta} \mathbb{E} \left[ U(2\alpha \varepsilon e^{rt} + Z^{\nu^\varepsilon}_t) - U(2\alpha \varepsilon e^{rt} + Z^\nu_t) \right].
\]

Using (5.13) together with the increase and the concavity of \( U \), this provides :

\[
\mathcal{J}^f_t(z, y; \nu) \leq \mathcal{J}^0_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) + e^{-\beta} \mathbb{E} \left[ U(2\alpha \varepsilon e^{rt} + Z^{\nu^\varepsilon}_t) - U(2\alpha \varepsilon e^{rt} + Z^\nu_t) \right] \\
\leq \mathcal{J}^0_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) + \alpha \varepsilon (e^{(r-\delta)t} - 1) \mathbb{E} \left[ U(2\alpha \varepsilon e^{rt} + Z^{\nu^\varepsilon}_t) - U(2\alpha \varepsilon e^{rt} + Z^\nu_t) \right] \\
\leq \mathcal{J}^0_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) + \alpha \varepsilon (e^{(r-\delta)t} - 1) \mathbb{E} \left[ U(2\alpha \varepsilon e^{rt} + Z^{\nu^\varepsilon}_t) - U(2\alpha \varepsilon e^{rt} + Z^\nu_t) \right] \\
\leq \mathcal{J}^0_t(z + 2\alpha \varepsilon, y; \nu^\varepsilon) + \alpha \varepsilon (e^{(r-\delta)t} - 1) \mathbb{E} \left[ U(2\alpha \varepsilon e^{rt} + Z^{\nu^\varepsilon}_t) - U(2\alpha \varepsilon e^{rt} + Z^\nu_t) \right].
\]

Now, observe from (5.13) that \( \nu^\varepsilon \in \mathcal{A}^f(z + 2\alpha \varepsilon, y) \). Then :

\[
\mathcal{J}^f_t(z, y; \nu) \leq V^0_t(z + 2\alpha \varepsilon, y) + e^{-\beta} U(\alpha \varepsilon e^{rt}) = V_t(z + 2\alpha \varepsilon) + e^{-\beta} U(\alpha \varepsilon e^{rt}) ,
\]

where the last equality follows from Propositions 3.1 and 3.2. The required inequality (5.12) is obtained by sending \( \varepsilon \) to zero and using the continuity of the Merton value function \( \hat{V} \).
References


