FROM RAGS TO RICHES: ON CONSTANT PROPORTIONS INVESTMENT STRATEGIES

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This paper studies the performance of self-financing constant proportions trading strategies, i.e. dynamic asset allocation strategies that keep a fixed constant proportion of wealth invested in each asset in all periods in time. We prove that any self-financing constant proportions strategy yields a strictly positive exponential rate of growth of investor's wealth in a financial market in which prices are described by stationary stochastic processes and the price ratios are non-degenerate. This result might be regarded as being counterintuitive because any such strategy yields no increase of wealth under constant prices. We further show that the result also holds under small transaction costs, which is important for the viability of this approach, since constant proportions strategies require frequent rebalancing of the portfolio.

Keywords: constant proportions strategies, balanced investment strategies, unbounded growth, stationary markets

1. Introduction

The problem of optimal portfolio selection is central to any theory of investment in financial markets. While investors' objectives can be manifold, it is often useful to focus on certain optimality criteria as benchmarks. The theory of optimum-growth portfolio, or log-optimum investment, studies portfolio selection rules that maximize the logarithmic growth rate of investor's wealth; see e.g. Algoet and Cover [1] and the survey by Hakansson and Ziemba [9]. When a strictly positive rate of growth can be achieved, wealth asymptotically becomes unbounded and, in the long run, overtakes any other investment strategy. The log-optimum investment principle—often referred to as the Kelly rule, Kelly [11], in the case of independent and identically distributed returns on investment—has proved quite successful in actual financial markets, Thorp [14].

One might be tempted to consider self-financing investment strategies that yield exponential growth of wealth as being exceptional and in general difficult to find—in particular if prices are given by stationary processes. However, as we show in this paper, *any* constant proportions trading strategy yields unbounded and exponentially fast growth of wealth in a stationary financial market, provided the investor trades in at least two stocks. To derive this result we need only a mild assumption of non-degeneracy of the price process. By definition, constant proportions strategies require the investment of a fixed constant proportion of wealth in each asset in all periods in time. These trading strategies are self-financing and only call for a non-zero initial investment; hence investors following this rule go “from rags to riches.” This result seems counterintuitive because any constant proportions strategy yields no increase of wealth under constant prices. Stationarity of the financial market rules out any systematic gain from investments, e.g. through increasing prices. However, any persistent stationary variation of prices (not being identical over assets) yields unbounded growth of wealth under every constant proportions strategy. Note that our assumption of stationarity of prices also implies that any buy-and-hold investment strategy does not yield an increase of wealth on average.

Constant proportions strategies have been studied—inspired by the optimality properties of the Kelly rule—in many different frameworks, see e.g. Browne and Whitt [7], Browne [6], Aurell et al. [3], and Aurell and Muratore-Ginanneschi [4].

The issue of transaction costs is quite important for the analysis of optimal investment. In our approach, transaction costs disclose the major drawback of constant proportions strategies—the frequent rebalancing of the portfolio. We take this criticism into account by showing that our result also holds when transactions are costly—provided the costs are sufficiently small.

In a related model with one riskless and one risky asset, Aurell and Muratore-Ginanneschi [4] studied the problem of optimal investment in the presence of transaction costs. In a continuous time framework in which the relative price is given by a diffusion process they show that the optimal investment is a constant proportions strategy with a friction in rebalancing. Only if the distribution of wealth differs significantly from the proportions prescribed by the Kelly rule in the absence of transaction costs should an investor rebalance his portfolio. The authors derive an explicit formula for the optimal investment rule by an approximation of the optimization problem. Serva [13] also founds that it is optimal not to continuously modify a constant proportions portfolio in the presence of transaction costs. He illustrates numerically with NYSE index data that there is an optimal lag for rebalancing.

In our paper a substantial role is played by the notion of a balanced investment strategy. This notion (in a somewhat different form) was first employed in the context of stochastic generalizations of the von Neumann economic growth model by Radner [12]. It was analyzed in a quite general setting by Arnold et al. [2]. This approach appears to be new to the financial market literature.

Due to the partial equilibrium character of our analysis, the impression of a money machine might arise and, moreover, it might be conjectured that constant proportions strategies are uninteresting when closing the model and dealing with general equilibrium. This perspective has recently been explored from an evolutionary point of view in a different strand of literature, Blume and Easely [5], Hens and Schenk–Hoppé [10], and Evstigneev et al. [8]. Their findings emphasize the relevance of constant proportions investment strategies in an equilibrium model.

The next section explains the model without transaction costs. Section 2 presents the main result on unbounded growth of wealth. The model with transaction costs is introduced and analyzed in Section 3.

2. The model

Let an investor observe prices and take actions in discrete periods of time $t = 0, 1, 2, \ldots.$ The factors underlying price changes are uncertain, and they are described in probabilistic terms. Uncertainty is modelled by a stochastic process s_t, $t = 0, \pm 1, \pm 2, \ldots$, taking values in a measurable space S. The value of the random parameter s_t characterizes the "state of the world" at time t.

Consider a financial market with $K \geq 2$ assets whose prices $p_t > 0$, $t = 0, 1, 2, \ldots$, form a sequence of strictly positive random vectors with values in the K-dimensional linear space R^K. We assume that p_t depends on the history of the process s_t up to time t, i.e.

$$p_t = p_t(s^t),\ s^t = (\ldots, s_{t-1}, s_t).$$

(All functions of s^t considered in what follows are supposed to be measurable.)

At each time period t, an investor chooses a *portfolio* $h_t(s^t) = (h_t^1(s^t), \ldots, h_t^K(s^t)) \geq 0$, where h_t^i is the number of units of asset i in the portfolio h_t. The assumption of non-negativity of h_t rules out short sales of the assets in our model. A sequence $h_t(s^t)$, $t = 0, 1, 2, \ldots$, specifying a portfolio at each time period t and in every random situation s^t, is called a *trading strategy.*

We begin with an analysis of the case of no transaction costs; then we describe changes that are necessary for dealing with situations where transactions are costly.

Given a number $w_0 > 0$, we say that h_t, $t = 0, 1, 2, ...$, is a trading strategy with *initial wealth* w_0 if $p_0\, h_0 = w_0$. A trading strategy is termed *self-financing* if

$$p_t(s^t)\, h_t(s^t) \leq p_t(s^t)\, h_{t-1}(s^{t-1}), \quad t = 1, 2, ... \quad \text{(a.s.)}. \tag{2.1}$$

The inequalities in (2.1) are supposed to hold almost surely (a.s.) with respect to the probability measure P induced by the stochastic process s_t, $t = 0, \pm 1, \pm 2, ...$, on the space of its paths. These inequalities state that the budget constraint, imposing restrictions on the choice of the investor's portfolio in every time period, is determined by the value of the previous period's portfolio at the current prices.

Let us say that the market is *stationary* if the process s_t is stationary and the price vectors p_t do not explicitly depend on t, i.e., $p_t = p(s^t)$. When analyzing such markets, it is of interest to consider trading strategies of *balanced growth* (or, briefly, *balanced* strategies). These strategies are of the form

$$h_t(s^t) = \gamma(s^1) ... \gamma(s^t)\, \tilde{h}(s^t), \quad t = 1, 2, ..., \tag{2.2}$$

where $\gamma(\cdot) > 0$ is a scalar-valued function and $\tilde{h}(\cdot) \geq 0$ is a vector function such that $\ln \gamma(s^t)$ and $\ln |\tilde{h}(s^t)|$ are integrable with respect to the measure P (for a vector $h = (h^i)$, we write $|h| = \sum_i |h^i|$). In probabilistic terms, integrability of the above functions means finiteness of the expectations

$$E|\ln \gamma(s^t)| \ \text{ and } \ E|\ln |\tilde{h}(s^t)||. \tag{2.3}$$

In the case $t = 0$, we assume in (2.2) that

$$h_0(s^0) = \tilde{h}(s^0). \tag{2.4}$$

The term "balanced" used in the foregoing definition is justified because (2.2) implies that all ratios

$$\frac{h_t^i(s^t)}{h_t^j(s^t)} = \frac{\tilde{h}^i(s^t)}{\tilde{h}^j(s^t)},\ i \neq j, \tag{2.5}$$

describing the proportions between the amounts of different assets in the portfolio, form stationary stochastic processes. Furthermore, the random growth rate of the amount of each asset $i = 1, ..., K$, in the portfolio

$$\frac{h_t^i(s^t)}{h_{t-1}^i(s^{t-1})} = \gamma(s^t)\, \frac{\tilde{h}^i(s^t)}{\tilde{h}^i(s^{t-1})}$$

is a stationary process. Clearly, for a balanced strategy the self-financing condition (2.1) is equivalent to

$$\gamma(s^t)\, p(s^t)\, \tilde{h}(s^t) \leq p(s^t)\, \tilde{h}(s^{t-1}) \quad \text{(a.s.)}. \tag{2.6}$$

In view of stationarity, if (2.6) holds for some t, then it holds automatically for all t. Finally, if every component $p^k(s^t)$, $k = 1, 2, ..., K$, of the vector $p(s^t)$ satisfies

$$E|\ln p^k(s^t)| < \infty, \tag{2.7}$$

then we can associate a balanced strategy (2.2) to any non-negative vector function $\tilde{h}(s^t)$ with $E|\ln|\tilde{h}(s^t)|| < \infty$ by defining

$$\gamma(s^t) := \frac{p(s^t)\,\tilde{h}(s^{t-1})}{p(s^t)\,\tilde{h}(s^t)}. \tag{2.8}$$

For this strategy, relations (2.1) and (2.6) hold as equalities.

In the sequel, we will assume that *the process* s_t, $t = 0, \pm 1, \pm 2, ...,$ *is ergodic, and the prices* $p^1(s^t)$, ..., $p^K(s^t)$ *satisfy (2.7).*

Our analysis will be based on the following result. The proposition below shows that, under quite general assumptions, the growth rate of wealth of any investor employing a balanced trading strategy is completely determined by the expected value of γ. We further show that strict positivity of $E\ln\gamma(s^0) \equiv E\ln\gamma(s^t)$ implies exponential growth of wealth, i.e., $p_t\,h_t \to \infty$ a.s. exponentially fast.

Proposition 1 *For any balanced trading strategy (2.2), we have*

$$\lim_{t\to\infty}\frac{1}{t}\ln(p_t\,h_t) = \lim_{t\to\infty}\frac{1}{t}\ln|h_t| = E\ln\gamma(s^0) \quad (a.s.). \tag{2.9}$$

Proof. We can write

$$\frac{1}{t}\ln|h_t| = \frac{1}{t}\sum_{m=1}^{t}\ln\gamma(s^m) + \frac{1}{t}\ln|\tilde{h}(s^t)|,$$

and so the second equality in (2.9) is an immediate consequence of the Birkhoff ergodic theorem, since $t^{-1}\ln|\tilde{h}(s^t)| \to 0$ by virtue of integrability of $\ln|\tilde{h}(s^t)|$. The first equality in (2.9) follows from the relations $|\ln(p_t\,h_t) - \ln|h_t|| \le \sum_k |\ln p^k(s^t)|$ and (2.7). □

The question which appears naturally when dealing with the above model is whether, in the present *stationary* context, there exist balanced strategies exhibiting possibilities for unbounded growth. Consider, for the moment, the deterministic case, where S consists of a single point. Then the self-financing condition (2.1) reduces to

$$p\,h_t \le p\,h_{t-1}, \tag{2.10}$$

with some constant price vector $p > 0$ (in the deterministic case, a stationary process is nothing but a constant). A balanced strategy is given by $h_t = \gamma^t\tilde{h}$, where $\gamma > 0$ is a constant scalar and $\tilde{h} \ge 0$ is a constant non-zero vector.

We can immediately see from (2.10) that the maximum possible value for γ is 1, which rules out any possibility of a non-zero growth.

The above deterministic argument totally agrees with our intuition, and it would be natural to expect that it could be extended to the general, stochastic case. However, this intuition fails, and it turns out that, in a stochastic world, one can usually design a variety of balanced strategies exhibiting almost surely unbounded, and even exponential, growth. Moreover, as the results in the next section show, the exponential growth is a typical phenomenon, which can be established for a broad class of balanced strategies described in terms of *proportional* investment rules. Since the prices of the assets form stationary processes, no dividends are paid, and the strategies we deal with are purely self-financing, this result may look, at the first glance, counterintuitive.

To formalize the idea of proportional investments, we introduce the following definition, which plays a key role in this paper. Let $\lambda = (\lambda_1, ..., \lambda_K)$ be a vector in the open simplex

$$\Delta^K = \left\{(\lambda_1, ..., \lambda_K) : \lambda_k > 0,\ \sum_k \lambda_k = 1\right\}.$$

We shall say that a trading strategy h_t is a *constant proportions strategy* if

$$p_t^k\, h_t^k = \lambda_k\, p_t\, h_{t-1} \tag{2.11}$$

for all $t = 1, 2, ...$ and $k = 1, ..., K$. In every period in time, an investor using this strategy rebalances her portfolio by investing the constant share λ_k of her wealth $p_t\, h_{t-1}$ into the kth asset. The wealth at the beginning of period t is determined by evaluating the portfolio h_{t-1} from the previous period at the current prices p_t^k. Note that a constant proportions strategy is always self-financing because (2.11) implies $p_t\, h_t = p_t\, h_{t-1}$. Also, note that such a strategy is uniquely defined by its initial portfolio $h_0(s^0)$ and the vector λ: for every $t \geq 1$, the portfolio $h_t(s^t)$ can be determined recursively by using equation (2.11). Therefore we shall say that h_t is *generated by* λ *and* h_0. In this paper, we deal only with those strategies for which $\lambda_k > 0$ for all k; proportional investment rules of this kind are sometimes termed *completely mixed.*

Consider two constant proportions strategies h_t and $\hat{h}_t$ generated by one and the same $\lambda \in \Delta^K$ and different initial portfolios $h_0 \neq 0$ and $\hat{h}_0 \neq 0$. It follows from (2.11) that

$$c\, h_t^k \leq \hat{h}_t^k \leq C\, h_t^k, \quad k = 1, ..., K,\ t = 1, 2, ..., \tag{2.12}$$

where c and C are defined by

$$c = \frac{|\hat{h}_0|}{|h_0|} \cdot \frac{\min_n p_1^n}{\max_n p_1^n}, \qquad C = \frac{|\hat{h}_0|}{|h_0|} \cdot \frac{\max_n p_1^n}{\min_n p_1^n},$$

with $p_1^n = p^n(s^1)$. Inequalities (2.12) can be established, by using (2.11), first for $t = 1$ and then for all $t > 1$ by way of induction. Relations (2.12) show that *asymptotic properties of constant proportions strategies do not depend on their initial portfolios.* This concerns, in particular, the property of exponential growth: we have $\lim t^{-1} \ln |h_t| > 0$ a.s. if and only if $\lim t^{-1} \ln |\hat{h}_t| > 0$ a.s.

3. The main result

In this section, we state and prove a central result of the paper (Theorem 1). To derive the result, we impose a condition of non-degeneracy of the price process $p(s^t)$: *with positive probability, the variable $p^k(s^t)/p^k(s^{t-1})$ is not constant with respect to $k = 1, 2, ..., K$, i.e., there exist m and n (that might depend on s^t) for which*

$$\frac{p^m(s^t)}{p^m(s^{t-1})} \neq \frac{p^n(s^t)}{p^n(s^{t-1})}. \tag{3.13}$$

Under this assumption, the following theorem holds.

Theorem 1 *Let $\lambda = (\lambda_1, ..., \lambda_K)$ be a vector in Δ^K, and let w_0 be a strictly positive number. Then there exists a vector function $h_0(s^0) \geq 0$ such that the constant proportions strategy h_t generated by λ and h_0 is a balanced strategy with initial wealth w_0, and we have*

$$\lim_{t\to\infty} \frac{1}{t} \ln p_t h_t = \lim_{t\to\infty} \frac{1}{t}\ ln|h_t| > 0 \quad (a.s.). \tag{3.14}$$

Thus, given any vector of proportions $\lambda \in \Delta^K$ and an initial wealth $w_0 > 0$, we can construct an initial portfolio h_0 satisfying the budget constraint $p_0\, h_0 = w_0$ so that the constant proportions strategy h_t defined recursively by (2.11) turns out to be balanced and exhibits exponential growth. Recall that this strategy is automatically self-financing by virtue of (2.11). As has been noticed at the end of the previous section, the property of exponential growth of h_t does not depend on the initial portfolio, and will retain if we replace h_0 by any other $\hat{h}_0 \neq 0$. (However, the strategy h_t might become non-balanced under such a replacement).

Proof of Theorem 1. Define

$$\tilde{h}(s^t) = \Big(\frac{\lambda_1\, w_0}{p^1(s^t)}\,,\ ...\,,\ \frac{\lambda_K\, w_0}{p^K(s^t)}\Big) \tag{3.15}$$

and

$$\gamma(s^t) = \frac{p(s^t)\, \tilde{h}(s^{t-1})}{w_0} \quad \left[= \sum_{k=1}^{K} \lambda_k\, \frac{p^k(s^t)}{p^k(s^{t-1})}\right]. \tag{3.16}$$

By virtue of (2.7), the expectations (2.3) are finite. Consider the balanced strategy h_t defined by (2.2) and (2.4), where $\tilde{h}$ and γ are specified in (3.15), (3.16). For each $t = 0, 1, ...,$ we have

$$\begin{aligned} p^k(s^{t+1})\, h^k_{t+1}(s^{t+1}) &= p^k(s^{t+1})\, \gamma(s^1) \ldots \gamma(s^{t+1})\, \frac{\lambda_k\, w_0}{p^k(s^{t+1})} \\ &= \gamma(s^1) \ldots \gamma(s^t) \cdot \gamma(s^{t+1})\, \lambda_k\, w_0 \\ &= \gamma(s^1) \ldots \gamma(s^t) \cdot p(s^{t+1})\, \tilde{h}(s^t)\, \lambda_k = \lambda_k\, p(s^{t+1})\, h_t(s^t). \end{aligned}$$

Thus h_t coincides with the constant proportions strategy generated by $\lambda = (\lambda_1, ..., \lambda_K)$ and $h_0 = \tilde{h}(s^0)$, and, furthermore, $p(s^0)\, h_0(s^0) = p(s^0)\, \tilde{h}(s^0) = \sum_k \lambda_k w_0 = w_0$.

In view of Proposition 1, it remains to show that $E \ln \gamma(s^t) > 0$. By virtue of Jensen's inequality (which is applicable since $\lambda_k > 0$ and $\sum \lambda_k = 1$), we have

$$\ln \sum_{k=1}^{K} \lambda_k\, \frac{p^k(s^t)}{p^k(s^{t-1})} \geq \sum_{k=1}^{K} \lambda_k\, \ln \frac{p^k(s^t)}{p^k(s^{t-1})}, \tag{3.17}$$

and, in view of assumption (3.13), the probability that this inequality is strict is greater than zero. The number $E \ln \gamma(s^t)$ is equal to the expected value of the expression on the left-hand side of (3.17). Consequently,

$$E \ln \gamma(s^t) > \sum_{k=1}^{K} \lambda_k\, E \ln \frac{p^k(s^t)}{p^k(s^{t-1})} \;=\; 0$$

by virtue of stationarity of s_t and finiteness of $E|\ln p^k(s^t)|$. □

Let us explain the intuition behind this result. Any constant proportions strategy 'exploits' the persistent fluctuation of prices in the following way. Keeping a fixed fraction of wealth invested in each asset implies that after a change in prices an investor sells those assets that are expensive relative to the other assets and purchases relatively cheap assets. The stationarity of prices implies that this portfolio rule yields a strictly positive expected rate of growth, despite the fact that each asset price has growth rate zero. Thereby investors go "from rags to riches."

4. Transaction costs

This section extends the result of the previous section to markets with transaction costs. Transaction costs represent the main obstacle in getting "from rags to riches" when using constant proportions strategies. This is due to the fact that their main disadvantage—the need to frequently rebalance the portfolio—becomes apparent when rebalancing is costly. However, since investors' wealth grows exponentially fast in the absence of transaction costs—as shown in Theorem 1, there should be room for small losses in every period (resulting from transactions) without eliminating the possibility of unbounded growth of wealth.

When transaction costs are present in the financial market, the self-financing condition (2.1) becomes

$$p_t h_t + \sum_{k=1}^{K} \delta_k p_t^k |h_t^k - h_{t-1}^k| \leq p_t h_{t-1} \quad (a.s.) \tag{4.18}$$

for all $t = 1, 2, ...$ (We write $p_t^k = p^k(s^t)$ and $h_t^k = h_t^k(s^t)$.) According to (4.18), the cost of transactions involving asset k is a fixed fraction $\delta_k \geq 0$ of the order volume. For simplicity of presentation, we assume that these fractions are the same for buying and selling.

We generalize the previous definition (2.11) by calling a trading strategy h_t a *constant proportions strategy* associated with a vector $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta^K$ if

$$p_t^k h_t^k = \lambda_k \left[p_t h_{t-1} - \sum_{n=1}^{K} \delta_n p_t^n |h_t^n - h_{t-1}^n| \right] \quad (a.s.) \tag{4.19}$$

for all $t = 1, 2, ...$ and $k = 1, 2, ..., K$. By virtue of this definition, the investment (evaluated in terms of the market price p_t) in asset k in every time period is equal to the fraction λ_k of the beginning-of-period wealth $p_t h_{t-1}$ less the total transaction costs. Clearly (4.19) implies (4.18), so that any strategy satisfying (4.19) is self-financing.

If there are no transaction costs, i.e. $\delta_n = 0$ for all $n = 1, ..., K$, then (4.19) coincides with equation (2.11). The latter equation allows to construct a constant proportions strategy recursively, based on the knowledge of λ and h_0. A similar construction can be performed when transaction costs are present as the proposition below shows.

Proposition 2 *Let $\lambda \in \Delta^K$, and let $\delta_1, ..., \delta_K$ be nonnegative numbers such that $\delta_k < 1$ for all k. Then, for every $h_{t-1} \geq 0$, there is a unique vector $h_t = (h_t^1, ..., h_t^K)$ satisfying (4.19). For this vector, we have*

$$h_t^k = \frac{\lambda_k}{p_t^k} \beta, \tag{4.20}$$

where β is a unique non-negative solution to the equation

$$\beta + \sum_{n=1}^{K} \delta_n |\lambda_n \beta - p_t^n h_{t-1}^n| = p_t h_{t-1}. \tag{4.21}$$

Consequently, in the model with transaction costs $\delta_k < 1$ as well as in the model without transaction costs, we can speak of constant proportions strategies *generated* by a vector of proportions λ and an initial portfolio h_0.

Before proving the above proposition, we formulate the main result of this section.

Theorem 2 *Let $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta^K$, and let $w_0 > 0$. Then there exists a number $\varepsilon > 0$ for which the following assertion holds. In the model with transaction costs $\delta_1, ..., \delta_K \geq 0$ not exceeding ε, one can construct a portfolio $h_0(s^0) \geq 0$ such that the constant proportions strategy h_t generated by λ and h_0 is a balanced strategy with initial wealth w_0, growing with an exponential rate: $\lim_{t\to\infty} t^{-1} \ln p_t h_t = \lim_{t\to\infty} t^{-1} \ln |h_t| > 0$ (a.s.).*

The balanced portfolio rule in Theorem 2 can be calculated numerically with minimal effort. Only the solution to equation (4.21) has to be determined in each period in time to obtain the constant proportions strategy. The growth rate of wealth is given by the expected value of the logarithm of the solution to (4.21).

The theorem says that conclusions analogous to those obtained in the previous section can be established for a model with transaction costs, provided these costs are small enough. We prove Theorem 2 by using a "small perturbation" technique, showing that the original model may be viewed as a limit of the perturbed one as $\delta = (\delta_1, ..., \delta_K) \to 0$. We prove Proposition 2 before turning to Theorem 2.

Proof of Proposition 2. We can see that $h_t = (h_t^k)_{k=1,...,K} \geq 0$ is a solution to (4.19) if and only if, for some $\beta \geq 0$, relations (4.20) and (4.21) hold. Thus it suffices to show that (4.21) has a unique solution $\beta \geq 0$. Denote the expressions on the left-hand side and on the right-hand side of (4.21) by $f(\beta)$ and α, respectively. The function $f(\beta)$ is continuous on $[0, \infty)$, and we have $f(0) \leq \alpha$ (since $\delta_k < 1$) whereas $f(\beta) > \alpha$ for all β large enough. Therefore at least one solution of the equation $f(\beta) = \alpha$ exists. To verify that this solution is unique observe that $f(\beta)$ is strictly increasing on $(0, \infty)$. Indeed, f is continuously differentiable in β at all points in $(0, \infty)$ except for $\beta = p_t^n h_{t-1}^n / \lambda_n$, $n = 1, ..., K$, and at all points at which the derivative exists, it is bounded below by $1 - \sum_n \lambda_n \delta_n > 0$. $\square$

Remark 1 In the proof of Theorem 2 we will need the following inequalities for the solution β to the equation (4.21):

$$\min_n (p_t^n \, h_{t-1}^n) \leq \beta \leq p_t \, h_{t-1}. \tag{4.22}$$

The latter inequality is an immediate consequence of (4.21). Suppose the former is not true, i.e., $p_t^n h_{t-1}^n > \beta$ for all n. Then (4.21) yields

$$\beta - \sum_{n=1}^{K} \delta_n \, (\lambda_n \, \beta - p_t^n \, h_{t-1}^n) = p_t \, h_{t-1},$$

and so

$$\beta = \frac{\sum(1-\delta_n)p_t^n h_{t-1}^n}{1-\sum \delta_n \lambda_n} = \frac{\sum(1-\delta_n)p_t^n h_{t-1}^n}{\sum(1-\delta_n)\lambda_n} \geq \min_n \frac{p_t^n h_{t-1}^n}{\lambda_n} \geq \min_n (p_t^n h_{t-1}^n),$$

which proves the first inequality in (4.22).

Proof of Theorem 2. Assume $\delta_k < 1$, $k = 1, 2, ..., K$. As in the proof of Theorem 1, we set $\tilde{h}(s^t) = (w_0 \lambda_k / p^k(s^t))_{k=1,...,K}$. Further, we define $\gamma = \gamma(s^t)$ as the non-negative solution to the equation

$$\gamma + \sum_{n=1}^{K} \delta_n \, |\lambda_n \, \gamma - p_t^n \, \frac{\tilde{h}_{t-1}^n}{w_0}| = p_t \, \frac{\tilde{h}_{t-1}}{w_0}, \tag{4.23}$$

where $p_t^n = p^n(s^t)$ and $\tilde{h}_t = \tilde{h}(s^t)$. By virtue of Proposition 2, this solution exists and is unique. (If $\delta_n = 0$ for all $n = 1, ..., K$, then $\gamma(s^t)$ coincides with the function in (3.16).) From (2.7), we can see that $E|\ln|\tilde{h}(s^t)|| < \infty$. By virtue of inequalities (4.22), we have

$$\min_n \Big[\lambda_n \, \frac{p^n(s^t)}{p^n(s^{t-1})}\Big] \leq \gamma(s^t) \leq \sum_n \lambda_n \frac{p(s^t)}{p(s^{t-1})}, \tag{4.24}$$

and so $E|\ln \gamma(s^t)| < \infty$. Thus the functions $\tilde{h}(\cdot)$ and $\gamma(\cdot)$ just constructed define a balanced strategy $h_t = \gamma(s^1)...\gamma(s^t)\,\tilde{h}(s^t)$, $h_0 = \tilde{h}(s^0)$.

Let us prove that h_t coincides with the constant proportions strategy generated by $h_0 = \tilde{h}(s^0)$ and λ. To this end we have to verify (4.19), or, equivalently, to show that (4.20) and (4.21) hold for some β. Define $\beta = \gamma(s^1)...\gamma(s^t)w_0$. Then we have

$$h_t^k = \gamma(s^1) \ldots \gamma(s^t) \, \tilde{h}^k(s^t) = \gamma(s^1) \ldots \gamma(s^t) \, \frac{w_0 \, \lambda_k}{p^k(s^t)} = \frac{\lambda_k}{p_t^k} \, \beta,$$

which proves (4.20). Finally, (4.21) is equivalent to the equation

$$\gamma(s^t) \, w_0 + \sum_{n=1}^{K} \delta_n \, |\lambda_n \, \gamma(s^t) \, w_0 - p_t^n \, \tilde{h}_{t-1}^n| = p_t \, \tilde{h}_{t-1},$$

holding by virtue of (4.23).

Note that the function γ involved in the construction of the balanced strategy h_t depends on the vector of transaction costs δ. For the sake of clarity, let us denote it by γ_δ. By the definition of $\gamma_\delta(s^t)$ (see (4.23)), we have $\gamma_\delta(s^t) \to \gamma_0(s^t)$ as $\delta \to 0$. It follows from (4.24) and (2.7) that there exists an integrable function $\theta(s^t)$ such that $|\ln \gamma_\delta(s^t)| \leq \theta(s^t)$. Therefore we can apply Lebesgue's dominated convergence theorem, which yields

$\lim_{\delta \to 0} E \ln \gamma_\delta(s^t) = E \ln \gamma_0(s^t)$. From Theorem 1 we know that, for $\delta = 0$, $E \ln \gamma_0(s^t) > 0$. Consequently, $E \ln \gamma_\delta(s^t) > 0$ for all δ such that $|\delta| < \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. To complete the proof it suffices to employ Proposition 1. $\square$

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