Money and prices under uncertainty

Tomoyuki Nakajima\textsuperscript{2} \hspace{1cm} Herakles Polemarchakis\textsuperscript{3}

May 2, 2003\textsuperscript{4}

\textsuperscript{1}We had very helpful conversations with Jayasri Dutta, Ronel Elul, John Geanakoplos, David Kelsey, Demetre Tsomokos and Michael Woodford; an editor and referees made valuable comments and suggestions. The work of Polemarchakis would not have been possible without earlier joint work with Gaetano Bloise and Jacques Drèze.

\textsuperscript{2}Department of Economics, Brown University; tomoyuki_nakajima@brown.edu

\textsuperscript{3}Department of Economics, Brown University; herakles_polemarchakis@brown.edu

Abstract

Monetary policy does not suffice to determine the stochastic path of inflation. A “nominal equivalent martingale measure”, associated with nominal asset prices, and the initial price level characterize the indeterminacy of monetary equilibria under uncertainty; when prices are flexible, the “nominal equivalent martingale measure” determines the distribution of rates of inflation up to the first moment. Monetary policy sets interest rates or money supplies; fiscal policy is Ricardian or non-Ricardian; asset markets are complete or incomplete; in commodity markets, competition may be imperfect and prices sticky; the asset market opens before or after the commodity market at each date-event. Recursive equilibria under interest rate policy are, also, indeterminate. Indeterminacy, here, does not derive from the stability of a deterministic steady state. Determinacy of the real allocation requires that the asset market be complete, monetary policy set interest rates, prices be flexible and the exchange of assets precede the exchange of commodities.

Key words: monetary policy; fiscal policy; financial policy; sticky prices; monopolistic competition; uncertainty; indeterminacy.

JEL classification numbers: D50; D52; E31; E40; E50.
1 Introduction

Whether monetary policy can control the inflation rate is a question of theoretical interest and practical importance. For example, the failure to control inflation can be the cause of suboptimal fluctuations if indeterminacy is real. Optimal fiscal-monetary policy supports an optimal allocation of resources; if such a policy is also consistent with other, suboptimal, equilibrium allocations, then it does not “implement” the targeted allocation.\(^1\) The possible role of fiscal policy in price-level determination depends on the extent to which monetary policy determines the inflation rate.\(^2\)

To address this question, we consider a stochastic cash-in-advance economy in which monetary policy sets interest rates or money supplies. We show that monetary policy leaves indeterminacy indexed by the initial price level and a “nominal equivalent martingale measure”. The nominal equivalent martingale measure is a measure associated with nominal asset prices; when prices are flexible, it determines the distribution of rates of inflation, up to the first moment, which is determined by the risk-free rate and arbitrage. In a finite-horizon economy, the degree of indeterminacy equals the number of the terminal nodes in the date-event tree. This result holds regardless of whether prices are flexible or sticky; whether monetary policy sets nominal interest rates or money supplies; whether or not the asset market opens before the goods market. The indeterminacy is real unless prices are flexible, monetary policy sets nominal interest rates, the asset market is complete, and the asset market opens before the goods market at each date-event.

A two-period economy where money only serves as a unit of account illustrates that monetary policy does not determine the path of the inflation rate under uncertainty. Let the state of the world in the second period be \(s = 1, \ldots, S\). There is a single commodity at each date-event. As is well known, the nominal prices of commodities, \(P(0)\) and \(\{P(s)\}\), are not determined: if \(\{q(s)\}\) is the nominal state price, only the discounted contingent relative prices, \(\{q(s)P(s)/P(0)\}\), are determined in equilibrium. In other words, there are \((S + 1)\) degrees of (nominal) indeterminacy in such an economy.\(^3\) The question here is whether this indeterminacy is reduced by an operative money market and monetary policy. The answer is that the reduction of the indeterminacy is only of one degree: there still remain \(S\) degrees of indeterminacy: the only restriction that interest rate policy, \(r(0), \{r(s)\}\), imposes on the nominal state prices is that

\[
\sum_s q(s) = \frac{1}{1 + r(0)}
\]

This equation shows that nominal state prices satisfy \(q(s) = \mu(s)/(1 + r(0))\), for some strictly

---

\(^1\)A useful survey of this literature is Chari and Kehoe (1999).

\(^2\)See, for example, Woodford (1996, 1999), Benhabib, Schmitt-Grohé and Uribe (2001, 2002).

\(^3\)Following Cass (1985), Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989) showed that, when the asset market is incomplete, such indeterminacy has real effects of dimension \((S - 1)\) or \((S - A)\) if the rates of return on assets are held fixed.
positive probability measure $\mu$, which is referred to as a “nominal equivalent martingale measure”. The indeterminacy is then characterized by the initial price level $P(0)$ and the nominal equivalent martingale measure $\mu$. Whether or not indeterminacy is real depends on the conduct of monetary policy, the completeness of the asset market and the order in which the asset and goods markets open.

Perhaps surprisingly, the argument extends to economies with monopolistic competition and sticky prices: both the initial price level and the nominal equivalent martingale measure are indeterminate, regardless of the degree of competition or the flexibility of prices in commodity markets, and the degree of indeterminacy is exactly the same as in the economy with perfect competition and flexible prices. This is so because predetermined prices effectively remove the same number of supply conditions. That indeterminacy is real when prices are sticky is due to the fact that output is “demand-determined.”

The fact that monetary policy leaves indeterminacy of degree equal to the number of the terminal nodes of the date-event tree is the reason that fiscal policy may matter for price level determination. Following Woodford (1996, 1999) and Benhabib, Schmitt-Grohé and Uribe (2001, 2002), a fiscal policy rule is called Ricardian if it implies that the public debt vanishes at each terminal node for all possible, equilibrium or non-equilibrium, values of price levels and other endogenous variables. Such a fiscal policy rule does not matter for price level determination. If, however, the fiscal policy is not of this form, it may add additional restrictions on equilibrium price levels, because public debt at each terminal node must be vanish in equilibrium, due to the transversality condition of households. If, for example, fiscal policy sets the composition of the debt portfolio and the level of real transfers at each date-event, then it is non-Ricardian, and the indeterminacy we discuss here vanishes.

It is worth emphasizing that the type of indeterminacy in our paper does not derive from the stability of a steady state or the infinity of the horizon. For this reason, we state most of the results for finite-horizon economies. Nevertheless, one might want to restrict attention to those equilibria that stay in a neighborhood of a steady state. In such a case, its stability would be important, which is related to recent discussions of the Taylor rule: although, as long as fiscal policy is Ricardian, the coefficient in the Taylor rule does not change the degree of indeterminacy, it affects the number of locally bounded equilibria.

There is a vast literature on indeterminacy of monetary equilibria. One branch of this literature is in macroeconomics: Sargent and Wallace (1975) discussed the indeterminacy of the initial price level under interest-rate policy; Stokey and Lucas (1987) derived the condition under which there exists a unique recursive equilibrium in a cash-in-advance economy with

---

4Carlstrom and Fuerst (1998).
5Benhabib and Farmer (1999) is a useful survey of the literature on indeterminacy arising from the stability of a steady state.
6Woodford (1999) and Benhabib, Schmitt-Grohé and Uribe (2001); Benhabib, Schmitt-Grohé and Uribe (2002) examine how non-Ricardian fiscal policy interacts with the Taylor rule to obtain a unique equilibrium.
money supply policy; Woodford (1994) analyzed the dynamic paths of equilibria associated with the indeterminacy of the initial price level under money supply policy. In this paper, we give the exact characterization of the indeterminacy in stochastic economies in terms of the initial price level and the nominal equivalent martingale measure and extend the argument to the sticky-price case; also, we show that there is a continuum of recursive equilibria with interest rate policy. Carlstrom and Fuerst (1998) discussed the indeterminacy of sticky-price equilibria when the nominal interest rates are zero. Here, we discuss the indeterminacy in more general case. Elsewhere, Carlstrom and Fuerst (2001) considered the money-in-the-utility model and compared the case in which the real balances at the beginning of each period are in the flow utility function and the case in which those at the end of each period are. Here we show a corresponding result in the cash-in-advance economy: we compare the economy in which the goods market opens before the asset market and the one in which the markets open in the reverse order. In closely related models, Dubey and Geanakoplos (1992, 2000) considered non-Ricardian fiscal policy with no transfers and Tsomokos (2001) extended their model to an open economy; Drèze and Polemarchakis (2000) and Bloise, Drèze and Polemarchakis (2003) studied the existence and indeterminacy of monetary equilibria with a particular Ricardian fiscal policy, seigniorage distributed contemporaneously as dividend to the private sector. Our paper is an application of these results to macroeconomic models.

The rest of the paper is organized as follows: In Section 2, we study a benchmark, two-period economy with flexible prices. In Sections 3 and 4, we extend the indeterminacy results to sticky-price economies. In Section 5, we consider an infinite-horizon economy. In Section 6, we discuss the implications of commodity markets that open before the asset market.

2 Flexible Prices

In this section, we describe the benchmark economy with flexible prices and we characterize the set of equilibria. All markets are perfectly competitive. Money is valued through a cash-in-advance constraint, as in Lucas and Stokey (1987). We show that the nominal equivalent martingale measure, as well as the initial price level, are indeterminate, and hence, the degree of indeterminacy is equal to the number of the terminal nodes of the date-event tree.

2.1 Households

There are three periods: \( t = 0, 1, 2 \). A stochastic shock, \( s \in S = \{1, \ldots, S\} \), realizes at the beginning of the second period. Each state occurs with a probability \( f(s) > 0 \). Production and consumption occur in the first two periods; the last period is added for accounting
purposes — households and the fiscal authority to redeem their debt.\textsuperscript{7}

There is a continuum of identical households, distributed uniformly over \([0, 1]\). At each
date-event, households produces a single, homogeneous product. The output produced by a
representative household in period 0 is \(y_0\) and at state \(s\) in period 1, it is \(y_1(s)\); consumption
is \(c_0\) and \(c_1(s)\).

The preferences of the representative household are described by the lifetime expected utility
\[
\begin{align*}
&u[c_0, \tilde{y}_0 - y_0] + \beta \sum_s u[c_1(s), \tilde{y}_1(s) - y_1(s)] f(s).
\end{align*}
\]

Here, we interpret \(y\) as the endowment of time, and \(y - y\) as the consumption of leisure, \(l\).\textsuperscript{8}

The flow utility function, \(u(c, l)\), satisfies standard conditions:

**Assumption 1.** The flow utility function, \(u : \mathbb{R}^2_{++} \to \mathbb{R}\), is continuously differentiable,
strictly increasing, and strictly concave. Both goods are normal:
\[
\begin{align*}
&u_{11}u_2 - u_{12}u_1 < 0, \quad \text{and} \quad u_{22}u_1 - u_{12}u_2 < 0.
\end{align*}
\]

The Inada conditions hold:
\[
\begin{align*}
&\lim_{c \to 0} u_1 = \lim_{l \to 0} u_2 = \infty.
\end{align*}
\]

In particular, this guarantees that \(u_1(c, y - c)/u_2(c, y - c)\) is strictly decreasing in \(c\).

We assume that a household cannot consume what it produces; instead, it has to purchase
consumption goods with cash from other households.\textsuperscript{9}

Concerning the timing of transactions we assume that at each date-event the asset market
opens before the goods market.\textsuperscript{10} An important consequence of this assumption is that the
cash the households obtains from sales of its output has to be carried over to the next period.

The representative household enters the initial period 0 with nominal assets \(w_0\). At the
beginning of the period, the fiscal authority distributes nominal transfers (taxes if negative),
\(\tau_0\), across households. Then, the asset market opens, in which cash and a complete set of
contingent claims are traded. The price of the contingent claim that pays off one unit of
currency if and only if state \(s\) occurs in the next period is \(q(s)\). The budget constraint for
the household in the asset market is
\[
\hat{m}_0 + \sum_s q(s)b_1(s) \leq w_0 + \tau_0,
\]

where \(\hat{m}_0\) is the amount of cash obtained by the household and \(b_1(s)\) the portfolio of ele-
mentary securities. With \(r_0\) the nominal interest rate, the no-arbitrage condition implies

\textsuperscript{7}Section 5 extends the argument to the infinite horizon case.

\textsuperscript{8}In the terminology of Lucas and Stokey (1987), \(y\) and \(y - y\) are the endowment and consumption of
“credit goods,” and \(c\) is consumption of “cash goods.”

\textsuperscript{9}In this, we follow Lucas and Stokey (1987).

\textsuperscript{10}See Section 6 for the case in which the goods market opens before the asset market.
that
\[ \sum_s q(s) = \frac{1}{1 + r_0}. \] (3)
The market for goods opens next. The purchase of consumption goods is subject to the cash-in-advance constraint
\[ P_0 c_0 \leq \hat{m}_0. \] (4)
The household also receives cash by selling its product, \( y_0 \). Hence, the amount of cash that it carries over to the next period, \( m_0 \), is
\[ m_0 = P_0 y_0 + \hat{m}_0 - P_0 c_0. \] (5)
Given (5), the cash-in-advance constraint (4) is equivalent to the constraint
\[ m_0 \geq P_0 y_0. \] (6)
The household enters state \( s \) in the second period with nominal wealth
\[ w_1(s) = m_0 + b_1(s). \] (7)
Substituting for \( \hat{m}_0 \) and \( b_1(s) \) from (5) and (7) into (2) yields the flow budget constraint in period 0:
\[ P_0 c_0 + \frac{r_0}{1 + r_0} m_0 + \sum_s q(s) w_1(s) \leq w_0 + \tau_0 + P_0 y_0. \] (8)
The household’s choice in the first period is subject to the flow budget constraint (8) and the cash constraint (6).

The transactions of the household in the second period are similar, except for the fact it faces no uncertainty. The nominal interest rate in state \( s \in S \) is \( r_1(s) \). The flow budget constraint and the cash constraint that the household faces at state \( s \) are
\[ P_1 c_1(s) + \frac{r_1(s)}{1 + r_1(s)} m_1(s) + \frac{1}{1 + r_1(s)} w_2(s) \leq w_1(s) + \tau_1(s) + P_1 y_1(s), \] (9)
and
\[ m_1(s) \geq P_1 y_1(s), \] (10)
where \( w_2(s) \) is the nominal wealth of the household at the end of state \( s \) in the second period.

In the following period, the only economic activity the household conducts is the repayment of its debt:
\[ w_2(s) \geq 0. \] (11)
Given this debt constraint, the flow budget constraints (8) and (9) reduce to the single, lifetime budget constraint

\[ P_0 c_0 + \frac{r_0}{1 + r_0} m_0 + \sum_s q(s) \left\{ P_1(s) c_1(s) + \frac{r_1(s)}{1 + r_1(s)} m_1(s) \right\} \leq w_0 + \tau_0 + P_0 y_0 + \sum_s q(s) \left\{ \tau_1(s) + P_1(s) y_1(s) \right\}. \] (12)

The cash constraints (6) and (10) take the form

\[ \frac{r_0}{1 + r_0} m_0 = \frac{r_0}{1 + r_0} P_0 y_0, \]
\[ \frac{r_1(s)}{1 + r_1(s)} m_1(s) = \frac{r_1(s)}{1 + r_1(s)} P_1(s) y_1(s), \]

because, if \( r > 0 \), the cash constraint binds; if \( r = 0 \) both sides of the equation are zero. Substituting these into the lifetime budget constraint, we obtain

\[ P_0 c_0 + \sum_s q(s) P_1(s) c_1(s) \]
\[ \leq w_0 + \tau_0 + \frac{P_1}{1 + r_0} y_0 + \sum_s q(s) \left\{ \tau_1(s) + \frac{P_1(s)}{1 + r_1(s)} y_1(s) \right\}. \] (13)

Given prices, \( P_0, P_1(s), r_0, r_1(s), \) and \( q(s) \), the household chooses \( c_0, c_1(s), y_0, \) and \( y_1(s) \) so as to maximize utility (1) subject to the life-time budget constraint (13). The lifetime budget constraint should bind at an optimum (the transversality condition); that is,

\[ w_2(s) = 0. \] (14)

The first-order necessary and sufficient conditions for a solution are

\[ \frac{u_1}{u_2} \left[ c_0, \overline{y}_0 - y_0 \right] = 1 + r_0, \]
\[ \frac{u_1}{u_2} \left[ c_1(s), \overline{y}(s) - y_1(s) \right] = 1 + r_1(s), \]
\[ \frac{\beta u_1}{u_1} \left[ c_1(s), \overline{y}(s) - y_1(s) \right] \frac{f(s)}{P_0} = q(s) P_1(s). \] (17)

According to (15)-(16), the marginal rate of substitution between consumption, \( c \), and leisure, \( \overline{y} - y \), equals the gross nominal interest rate; this is because production, \( y \), is taxed at the rate of nominal interest, due to the fact that the cash obtained from selling goods, should be carried over to the next period. Equation (17) is the standard Euler equation.
2.2 The public sector

With similar arguments, the flow budget constraints that the monetary-fiscal authority faces are

\[
\frac{r_0}{1 + r_0} M_0 + \sum_s q(s) W_1(s) = W_0 + T_0, \tag{18}
\]

\[
\frac{r_1(s)}{1 + r_1(s)} M_1(s) + \frac{1}{1 + r_1(s)} W_2(s) = W_1(s) + T_1(s), \tag{19}
\]

where \( M_0 \) and \( M_1(s) \) are money supplies, \( W_0, W_1(s), W_2(s) \) are the total liabilities of the monetary-fiscal authority, and \( T_0 \) and \( T_1(s) \) are aggregate transfers to the households.

**Monetary policy** Monetary policy sets either the nominal interest rates, \( r_0 \geq 0 \) and \( r_1(s) \geq 0 \) or money supplies, \( M_0 > 0 \) and \( M_1(s) > 0 \).

We assume that fiscal policy is “Ricardian”,\(^{11}\) and, in particular,\(^{12}\), with \( W_1(s) \in \mathcal{S} \) the “composition” of the debt portfolio of the public sector, and \( d \) its “scale;”

\[
W_1(s) = d W_1(s), \quad \text{and} \quad \sum_s W_1(s) = 1.
\]

**Fiscal policy** The fiscal authority sets the repayment rate \( \alpha \in (0, 1] \) and the composition of the debt portfolio, \( W_1(s) \). At date 0, given \( W_0, r_0, M_0 \) and \( \alpha \), the transfer, \( T(0) \), is determined by

\[
T(0) = \frac{r_0}{1 + r_0} M_0 - \alpha W_0,
\]

and the scale of debt portfolio, \( d \), is determined by the flow budget constraint

\[
d = \frac{1}{\sum_s q(s) W_1(s)} (1 - \alpha) W_0.
\]

At each state \( s \) in period 1, \( T_1(s) \) is set as

\[
T_1(s) = \frac{r(s)}{1 + r(s)} M_1(s) - W_1(s),
\]

where \( W_1(s) = d W_1(s) \).

This fiscal policy rule is Ricardian in that

\[
W_2(s) = 0, \tag{20}
\]

at all \( s \in \mathcal{S} \), and for all possible, equilibrium or non-equilibrium, values of \( P, r, \) and \( M \).

\(^{11}\)This is in the sense of Woodford (1996, 1999), Benhabib, Schmitt-Grohé and Uribe (2001, 2002), among others.

\(^{12}\)This is a stochastic analogue of the policy considered by Benhabib, Schmitt-Grohé and Uribe (2001, 2002).
2.3 Equilibrium conditions

Since households are identical, the market clearing conditions are

\[ c_0 = y_0, \quad c_1(s) = y_1(s), \]
\[ m_0 = M_0, \quad m_1(s) = M_1(s), \]
\[ w_1(s) = W_1(s), \quad w_2(s) = W_2(s). \]

Also, consistency requires that

\[ \tau_0 = T_0, \quad \tau_1(s) = T_1(s), \quad w_0 = W_0. \]

The no-arbitrage condition (3) implies that the prices of elementary securities, \( q(s), s \in S \), can be written as

\[ q(s) = \frac{\mu(s)}{1 + r_0}, \quad (21) \]

for some \( \mu(s), s \in S \), satisfying

\[ \sum_s \mu(s) = 1. \]

It follows that \( \mu \) is viewed as a probability measure over \( S \), and called the nominal equivalent martingale measure. We shall see that there are no equilibrium conditions that determine \( \mu \), regardless of whether monetary policy sets interest rates or money supplies. A competitive equilibrium with interest-rate policy is defined as follows:

**Definition.** Given initial nominal wealth, \( w_0 = W_0 \), interest-rate policy, \( \{r_0, r_1(s)\} \), and fiscal policy, \( \{\alpha, W_1(s)\} \), a competitive equilibrium consists of an allocation, \( \{c_0, c_1(s), y_0, y_1(s)\} \), a portfolio of households, \( \{m_0, m_1(s), w_1(s), w_2(s)\} \), a portfolio of the monetary-fiscal authority, \( \{M_0, M_1(s), W_1(s), W_2(s)\} \), transfers, \( \{T_0, T_1(s)\} \), spot-market prices, \( \{P_0, P_1(s)\} \) and a nominal equivalent martingale measure, \( \mu \), such that

1. given \( W_0 \) and \( \{r_0, r_1(s), M_0, M_1(s)\} \), fiscal policy \( \{\alpha, W_1(s)\} \) determines transfers \( \tau_0 = T_0 \) and \( \tau_1(s) = T_1(s), s \in S \), and debt portfolio \( \{W_1(s), W_2(s)\} \);
2. the monetary authority accommodates the money demand, \( M_0 = m_0 \) and \( M_1(s) = m_1(s) \);
3. given interest rates, \( r_0, r_1(s) \), spot-market prices, \( p_0(j) = P(0), p_1(s, j) = P_1(s) \), all \( j \), nominal equivalent martingale measure, \( \mu \), and transfers, \( \tau_0, \tau_1(s) \), the household’s problem is solved by \( c_0, c_1(s), y_0, y_1(s), m_0, m_1(s), w_1(s), \) and \( w_2(s) \);
4. all markets clear.

We restrict attention to symmetric equilibria.

A competitive equilibrium with money-supply policy is similarly defined.
2.4 Equilibria with interest rate policy

The existence of equilibrium with interest-rate policy requires further restrictions on the flow utility function of household.

**Assumption 2.** The flow utility function, $u$, satisfies

$$
\lim_{c \to 0} \frac{u_1(c, y - c)}{u_2(c, y - c)} = \infty,
$$

for each $y > 0$.

The following proposition shows that $P_0$ and $\mu$ are not determined, and hence, there is $S$-dimensional indeterminacy.

**Proposition 1.** Given initial nominal wealth, $w_0 = W_0$, interest-rate policy, $\{r_0, r_1(s)\}$, and fiscal policy, $\{\alpha, W_1(s)\}$, if Assumptions 1-2 are satisfied, then

1. a competitive equilibrium exists;
2. the equilibrium allocation $\{c_0, c_1(s), y_0, y_1(s)\}$ is unique;
3. the initial price, $P_0$, and the nominal equivalent martingale measure, $\mu$, are indeterminate: for any $P_0 > 0$ and for any strictly positive probability measure $\mu$, any prices and portfolios $\{P_1(s), M_0, M_1(s), W_1(s)\}$ that satisfy

$$
P_1(s) = \frac{\beta u_1(c_1(s), y_1(s) - y_0) f(s) 1 + r_0}{u_1(c_0, y_0) \mu(s)} \neq \infty,
$$

$M_0 \geq P_0 c_0, \quad M_1(s) \geq P_1(s) c_1(s), \quad (\text{with equality if } r_0, r_1(s) > 0),

$$
W_1(s) = (1 - \alpha)(1 + r_0) W_0 \frac{\sum_s \mu(s) W_1(s)}{\sum_s \mu(s) W_1(s)}
$$

support the allocation $\{c_0, c_1(s), y_0, y_1(s)\}$.

**Proof** Given interest rates $r_0$ and $r_1(s)$, $s \in S$, the first-order conditions (15)-(16) determines the allocation of resources at each date-event:

$$
\frac{u_1 [c_0, y_0 - c_0]}{u_2 [c_0, y_0 - c_0]} = 1 + r_0
$$

$$
\frac{u_1 [c_1(s), y_1(s) - c_0]}{u_2 [c_1(s), y_1(s) - c_1(s)]} = 1 + r_1(s).
$$

Our assumptions on $u$ guarantees the existence and uniqueness of the solutions to these equations. The equilibrium output at each date-event, $y_0$ and $y_1(s)$, $s \in S$, is given by

$$
y_0 = c_0, \quad \text{and} \quad y_1(s) = c_1(s).
$$
Fiscal policy sets transfers so that \( w_2(s) = W_2(s) = 0 \), all \( s \). Hence, the allocation is uniquely determined. It is straightforward to see that given any \( P_0 > 0 \), and \( \mu \), the prices and portfolio constructed as in the proposition support the equilibrium allocation.

\[ \square \]

The indeterminacy of \( \mu \) implies that the inflation rate, \( \pi_1(s) \equiv P_1(s)/P_0 \), is indeterminate. Thus, interest rate policy does not determine the stochastic path of inflation. The indeterminacy, which, here, is purely nominal, becomes real if, for example, prices are sticky (section 3-4); the asset market is incomplete; or the timing of the markets is different (section 6). Also shocks could be purely extrinsic. If \( r_1(s) \) and \( \tilde{\pi}_1(s) \) are identical for all \( s \), there is no uncertainty in “fundamentals;” nevertheless, there are equilibria in which the inflation rate, \( P_1(s)/P_0 \), varies across states.

The reason that \( P(0) \) and \( \mu \) are indeterminate is simple, and closely related to the well known fact that only relative prices are determined in equilibrium. Look at equation (12). The relative prices between consumption and real balances are \( r_0/[1 + r_0] \) and \( r_s/[1 + r_s] \), which are set by monetary policy. Given these prices, the equilibrium real balances, \( M_0/P_0 \) and \( M_1(s)/P_1(s) \), are determined. Also, the intertemporal relative prices of consumption, \( q(s)P_1(s)/P_0 \), are determined in equilibrium, which gives \( S \) restrictions on \( q(s), P_1(s), \) and \( P_0 \) (2S + 1 prices). In addition, the no-arbitrage condition (21) imposes one restriction on \( q(s) \). There are no further restrictions. Hence, there are \((S + 1)\) equations in \((2S + 1)\) variables, which leads to indeterminacy of degree \( S \), and \( P_0 \) and \( \mu \) are undetermined.

The fact that monetary policy leaves the indeterminacy whose degree equals the number of terminal nodes is the key to understand why certain forms of fiscal policy may lead to determinacy. Write the debt the public sector leaves at the end of the second period as a function of prices: \( W_2(s; P_0, \mu(s), P(s), s \in S) \). If the policy is Ricardian this is identically zero for any value of \( P_0, \mu(s), P(s) \). But if it is not, the equilibrium requirement that

\[ W_2(s; P_0, \mu(s), P(s), s \in S) = 0 \]

introduces \( S \) additional, nontrivial conditions on equilibrium prices. An example of such non-Ricardian policy considered by Woodford (1994, 1996, 1999) and Benhabib, Schmitt-Grohé and Uribe (2001, 2002), among others, is to set real transfers, \( T/P \), and the composition of the portfolio, \( \bar{W} \). Such fiscal policy eliminates the indeterminacy discussed here, and it is possible because monetary policy leaves the indeterminacy of degree \( S \).

Concerning the indeterminacy that obtains, two further remarks are in order:

1. The absence of real effects derives from the completeness of the asset market: with nominal assets and less than a full set of elementary securities or equivalent, different distributions of the rate of inflation across states of the world affect the attainable reallocations of revenue and, as a consequence, the allocation of resources in the presence of heterogeneous households. This is precisely the logic of the argument in the
literature on real indeterminacy with financial markets in the literature on general equilibrium with incomplete markets following Balasko and Cass (1989), Cass (1985) and Geanakoplos and Mas-Colell (1989). Importantly, the result there does not depend on the abstract specification without an explicit money market.

2. The portfolio policy of the public sector, the composition of the portfolio \( \{W_1(s)\} \), does not affect the allocation of resources at equilibrium; this is an instance of Ricardian equivalence.

2.5 Equilibria with money supply policy

Consider money supply policy, \( \{M_0, M_1(s)\} \). Define \( c^*_0 \) and \( c^*_1(s) \), implicitly by

\[
\frac{u_1[c^*, \bar{y}_0 - c^*_0]}{u_2[c^*, \bar{y}_1(s) - c^*_1(s)]} = 1.
\]

Such a \( c^* \) exists and is unique under Assumption 1; indeed, \( c^* \) is the level of consumption when the nominal interest rate is zero.

The existence of equilibrium with interest-rate policy requires alternative, further restrictions on the flow utility function of household.

**Assumption 3.** The flow utility function, \( u \), has the property that for all \( y > 0 \),

\[
\lim_{c \to 0} cu_1(c, y - c) = 0,
\]

and the function \( cu_1(c, y - c) \) is monotonically increasing in the interval \((0, c^*(y))\).

According to the proposition that follows, with money-supply policy there is the same degree of indeterminacy as with interest-rate policy — but indeterminacy is real.

**Proposition 2.** Given initial nominal wealth, \( w_0 = W_0 \), money-supply policy, \( \{M_0, M_1(s)\} \) and fiscal policy, \( \{\alpha, W_1(s)\} \), if Assumptions 1 and 3 are satisfied, then

1. a competitive equilibrium exists;

2. the initial price, \( P_0 \), and the nominal equivalent martingale measure, \( \mu \), are indeterminate. For any strictly positive \( P_0 \) and \( \mu \), there exists a unique competitive equilibrium corresponding to them.

3. the indeterminacy regarding \( P_0 \) and \( \mu \) is real: different \( P_0 \) or different \( \mu \) are associated with different allocations as well as different inflation rates.
Proof Let the initial price $P_0$ and the strictly positive probability measure $\mu$ be arbitrarily given. Let $M_0$ and $M_1(s)$ be the money supplies chosen by the policy. Given $M_0$ and $P_0$, $c_0$, $y_0$, and $r_0$ are determined by

$$c_0 = \min \left\{ \frac{M_0}{P_0}, r_0 \right\}, \quad 1 + r_0 = \frac{u_1[c_0, y_0 - c_0]}{u_2[c_0, y_0 - c_0]},$$

and $y_0 = y_0 - c_0$. At state $s$ in the second period, if

$$M_1(s) > \frac{\beta c_1(s)u_1[c_1(s), y_1(s) - c_1(s)]^2 f(s)}{u_2[c_0, y_0 - c_0]} \mu(s) P_0,$$

then let $c_1(s) = c_1^*(s)$. Otherwise, $c_1(s)$ is a solution to

$$M_1(s) = \frac{\beta c_1(s)u_1[c_1(s), y_1(s) - c_1(s)]^2 f(s)}{u_2[c_0, y_0 - c_0]} \mu(s) P_0.$$

The unique existence of a solution is guaranteed by Assumption 3. Given $c_1(s)$, $y_1(s) = y_1(s) - c_1(s)$,

$$P_1(s) = \frac{\beta u_1[c_1(s), y_1(s) - c_1(s)]^2 f(s)}{u_2[c_0, y_0 - c_0]} \mu(s) P_0,$$

and

$$1 + r_1(s) = \frac{u_1[c_1(s), y_1(s) - c_1(s)]^2}{u_2[c_1(s), y_1(s) - c_1(s)]^2} \theta - 1.$$ 

Given the path of nominal interest rates, $\{r_0, r_1(s)\}$, the debt portfolio, $\{W_1(s)\}$, is determined as in the proof of the previous proposition.

That money-supply policy leaves the same degree of indeterminacy as interest-rate policy is intuitive, because both types of monetary policy set the same number of variables (the former sets the supplies, $M$, and the latter sets the prices, $r$). The indeterminacy with money supply policy is real because the cash constraint implies that $Pc \leq M$ in equilibrium; different $P$ would be associated with different $c$.

Assumption 3 is made to simplify the proof, but from the construction of equilibria in the proof it is clear that the claim holds more generally.\(^{13}\) For example, define the set $A(s)$ by

$$A(s) = \{ c u_1[c, y(s) - c] : 0 < c \leq c*(s) \}.$$ 

If $A(s)$ has non-empty interior for all $s$, then the claim of the proposition holds with a slight modification. It is worth noting, however, that the log utility does not satisfy the non-empty interior condition. This is why, with the log utility, the equilibrium with strictly positive nominal interest rates is unique.

\(^{13}\)This is not the case in infinite horizon. See the discussion in Section 5.
Given recent discussions on the “liquidity trap,” the following corollary of the proposition would be of some interest.\textsuperscript{14} It says that, as long as money supply does not decrease too much in the second period, there always exists an equilibrium in which the nominal interest rate equals zero at all date-events.

\textbf{Corollary 3.} \textit{Given initial nominal wealth, }$w_0 = W_0$, \textit{interest-rate policy, }$\{r_0, r_1(s)\}$, \textit{and fiscal policy, }$\{\alpha, \overline{W}_1(s)\}$, \textit{if Assumption 1 is satisfied and, in addition,}

$$\frac{M_1(s)}{M_0} \geq \max_s \frac{\beta u_1[c^*_1(s), \overline{y}_1(s) - c^*_1(s)]}{\alpha u_2[c^*_0, \overline{y}_0 - c^*_0]},$$

\textit{then there exists a competitive equilibrium in which the nominal interest rate is identically zero, }$r_0 = r_1(s) = 0$, \textit{all }$s$.

\textbf{Proof} Choose any $P_0 \leq M_0/c^*_0$ and let $\mu = f$. Then, it is straightforward to see that the following allocation and price system constitute an equilibrium.

$c_0 = c^*_0, \quad c_1(s) = c^*_1(s),$

$y_0 = \overline{y}_0 - c^*_0, \quad y_1(s) = \overline{y}_1(s) - c^*_1(s),$

$P_1 = P_0 \frac{\beta u_1[c^*_1(s), \overline{y}_1(s) - c^*_1(s)]}{\alpha u_2[c^*_0, \overline{y}_0 - c^*_0]},$

and

$r_0 = r_1(s) = 0.$

3 Prices Set in Advance

We have seen that in a flexible price economy monetary policy does not determine an equilibrium and that the indeterminacy is indexed by the initial price level, $P_0$, and the nominal equivalent martingale measure, $\mu$. The question we address in this section is whether or not introducing price stickiness modifies those result.

Take the economy in the previous section and suppose that we are interested in an equilibrium with the property that the second-period prices, $P_1(s)$, are the same across states:

$P_1(s) = P_1, \quad s \in S.$

\textsuperscript{14}For example, Benhabib, Schmitt-Grohé and Uribe (2002) and Woodford (1999). Note, however, that in this model, a zero interest rate equilibrium is (constrained) optimal. A related fact that when a nominal interest rate is zero money supply is indeterminate is discussed by Carlstrom and Fuerst (1998) and Adao, Correia and Teles (2001).
Such an equilibrium could be viewed as a “sticky-price equilibrium” in that the second-period price, $P_1(s)$, does not depend on the realization of the shock, $s$, and hence it is “predetermined.” In the flexible-price economy, this additional restriction gets rid of indeterminacy with respect to $\mu$. To see this, consider interest-rate policy, and let $P_1$ be the price level in the second period, which does not depend on states $s$. Remember that interest-rate policy determines an allocation uniquely. Then the inflation rate, $P_1/P_0$, is uniquely determined by the condition that

$$\frac{P_1}{P_0} = \sum_s \frac{\beta u_1 [c_1(s), y_1(s) - c_1(s)] f(s)}{u_1 [c_1(s), y_1(s) - c_1(s)]}.$$ 

Given $P_1/P_0$, $\mu(s)$ is, in turn, determined by

$$\mu(s) = \frac{\beta u_1 [c_1(s), y_1(s) - c_1(s)] f(s) P_0}{u_2 [c_1(s), y_1(s) - c_1(s)] P_1}.$$ 

However, this only captures the uniqueness of “flexible-price equilibrium” with the property that the second-period price is identical across states. It does not imply the uniqueness of “sticky-price equilibrium” in which all price-setters explicitly take into account the constraint that the second-period price must be set in advance. Indeed, we shall see that the sticky-price equilibrium is indeterminate, that the degree of indeterminacy is exactly the same as the one associated with flexible-price equilibrium, and that the indeterminacy is real even under interest-rate policy.

As in the previous section, there is a continuum of households, which are distributed uniformly over $[0, 1]$. However, they differ what they produce: at each date-event, household $j \in [0, 1]$ produces a differentiated product $j$. Let $y_0(j)$ and $y_1(s, j)$ denote the amount of output produced by household $j$ in period 0 and at state $s \in S$ in period 1, respectively. The amount of commodity $i \in [0, 1]$ consumed by household $j$ is denoted by $c^j_0(i)$ and $c^j_1(s, i)$.

The lifetime utility of household $j$ is modified as

$$u[c^j_0, y_0 - y_0(j)] + \beta \sum_s u[c^j_1(s), y_1(s, j)] f(s),$$

where $c^j_0$ and $c^j_1(s)$ are consumption of the “composite” goods defined by

$$c^j_0 = \left\{ \int_0^1 [c^j_0(i)] \frac{\theta}{\theta - 1} di \right\}^{\theta - 1},$$

$$c^j_1(s) = \left\{ \int_0^1 [c^j_1(s, i)] \frac{\theta}{\theta - 1} di \right\}^{\theta - 1}.$$ 

Let $p_0(i)$ and $p_1(s, i)$ be the spot prices of good $i$ in period 0 and at state $s$ in period 1,
respectively. Then, the prices of the composite goods, \( P_0 \) and \( P_1(s) \), are given by

\[
P_0 = \left\{ \int_0^1 [p_0(i)]^{\frac{1}{1-\theta}} \, di \right\}^{1-\theta},
\]

\[
P_1(s) = \left\{ \int_0^1 [p_1(s, i)]^{\frac{1}{1-\theta}} \, di \right\}^{1-\theta}.
\]

The household’s cost minimization leads to

\[
P_0 c^0_j = \int_0^1 p_0(i) c^0_j(i) \, di,
\]

\[
P_1(s) c^1_j(s) = \int_0^1 p_1(s, i) c^1_j(s, i) \, di.
\]

Let \( c_0 \) and \( c_1(s) \) be the aggregate consumption at date 0 and state \( s \), that is,

\[
c_0 = \int_0^1 c^0_j \, dj, \quad \text{and} \quad c_1(s) = \int_0^1 c^1_j(s) \, dj.
\]

The demand for product \( j \) is then

\[
y_0(j) = \left( \frac{p_0(j)}{P_0} \right)^{-\theta} c_0, \quad (23)
\]

\[
y_1(s, j) = \left( \frac{p_1(s, j)}{P_1(s)} \right)^{-\theta} c_1(s). \quad (24)
\]

Suppose that the initial prices \( p_0(j), j \in [0, 1] \), are given and identical for all \( j \):

\[
p_0(j) = \overline{p}, \quad j \in [0, 1].
\]

It follows that \( P_0 = \overline{p} \). In the first period, each household \( j \in [0, 1] \) chooses the second-price, \( p_1(s, j) \), before observing the shock \( s \). It follows that the second-period price is identical across state, so that it is written as

\[
p_1(s, j) = p_1(j), \quad j \in [0, 1],
\]

for some \( p_1(j) \). Given \( \overline{p} \) and \( p_1(j) \), the household must supply product \( j \) by the amount equal to the demand:

\[
y_0(j) = \left( \frac{\overline{p}}{P_0} \right)^{-\theta} c_0, \quad (25)
\]

\[
y_1(s, j) = \left( \frac{p_1(j)}{P_1(s)} \right)^{-\theta} c_1(s). \quad (26)
\]

Given prices, \( P_0, P_1(s), r_0, r_1(s), \mu(s) \), and \( \overline{p} \), household \( j \) chooses \( c^0_j, c^1_j(s) \), and \( p_1(j) \) so as to maximize the lifetime expected utility (1) subject to the demand functions (25)-(26)

15
and the life-time budget constraint:

\[
P_0 c_0^j + \sum_s \frac{\mu(s)}{1 + r_0} P_1(s) c_1^j(s) \leq w_0 + \tau_0 + \frac{1}{1 + r_0} p_0(j)\text{.}
\]

The first-order conditions with respect to \(c_0\) and \(c_1(s)\) lead to

\[
\beta u_1 [c_1^j(s), \overline{y}_1(s) - y_1(s,j)] f(s) = P_1(s) \frac{\mu(s)}{1 + r_0}\text{.}
\]

The first-order condition with respect to \(p_1(j)\) is given by

\[
\frac{\sum_s u_1 [c_1^j(s), \overline{y}_1(s) - y_1(s,j)] f(s)}{\sum_s u_2 [c_1^j(s), \overline{y}_1(s) - y_1(s,j)] f(s)} = \frac{\theta}{\theta - 1}\text{,}
\]

where we have used the (symmetric) equilibrium condition that \(p_1(j) = P_1\), all \(j \in [0, 1]\).

The next proposition shows that the sticky-price economy has exactly the same degree of indeterminacy as the flexible-price economy, and that the indeterminacy is indexed by \(P_1\) and \(\mu\). Here, however, indeterminacy is real even with interest-rate policy.

**Proposition 4.** If second-period prices are set in advance, and if Assumptions 1 and 2 are satisfied, then, given initial nominal wealth, \(w_0 = W_0\) and price level, \(P_0 = p\), interest-rate policy, \(\{r_0, r_1(s), s \in S\}\), and fiscal policy, \(\{\alpha, W_1(s)\}\),

1. a competitive equilibrium exists;
2. the price level in the second period, \(P_1\), and the nominal equivalent martingale measure, \(\mu\), are indeterminate;
3. the indeterminacy is real: different \(P_1\) or different \(\mu\) are associated with different allocations.

**Proof** Let \(P_1\) and \(\mu\) be given. Then the first-order conditions (28) imply that equilibrium consumption, \(c_0\) and \(c_1(s)\) should satisfy

\[
\beta u_1 [c_1^j(s), \overline{y}_1(s) - c_1^j(s)] f(s) = \frac{P_1(s) \mu(s)}{P_0 1 + r_0}\text{.}
\]

Under our assumptions, these equations can be solved for \(c_1(s)\) as strictly increasing functions of \(c_0\). Write them as

\[
c_1(s) = \phi_s(c_0), \quad s \in S,
\]
where \( \lim_{c \to 0} \phi_s(c) = 0 \) and \( \lim_{c \to \overline{y}_0} \phi_s(c) = \overline{y}_s(\theta) \). The first-order condition (29) then implies that

\[
\sum_s u_1[\phi_s(c_0), \overline{y}_s(\theta) - \phi_s(c_0)] f(s)[1 + r_1(s)] = \frac{\theta}{\theta - 1}.
\]

Under our assumptions, there is a unique \( c_0 \) that satisfies this equation. This completes the proof.

The reason why the degree of indeterminacy remains the same is that fixing a certain number of prices is equivalent to removing the same number of supply conditions (with sticky prices output is determined by “demand”). To see this, it is illustrative to consider a little bit different form of sticky prices. Suppose that the initial prices, \( p_0(j), j \in [0, 1] \), are freely chosen, but that the second period prices are exogenously fixed:

\[
p_1(s, j) = P_1(s), \quad s \in S,
\]

where \( P_1(s) \) may vary across states. Note that if all prices are freely chosen, each household determines the level of output so that marginal disutility of labor equals marginal utility of consumption times the “effective” wage rate:

\[
u_2[c_0^j, \overline{y}_0 - y_0(j)] = u_1[c_0^j, \overline{y}_0 - y_0(j)] \frac{1}{1 + r_0} \frac{\theta - 1 p_0(j)}{P_0},
\]

\[
u_2[c_1^s, \overline{y}_1(\theta) - y_1(s, j)] = u_1[c_1^s, \overline{y}_1(\theta) - y_1(s, j)] \frac{1}{1 + r_1(s)} \frac{\theta - 1 p_1(s, j)}{P_1(s)}.
\]

Note that \( r/(1+r) \) is the tax rate on labor supply, \( \theta/(\theta - 1) \) is the markup (the wedge between marginal product and the real wage rate), and \( p(j)/P \) is the relative price of product \( j \). In a symmetric equilibrium, those supply conditions become

\[
u_2[c_0, \overline{y}_0 - c_0] = u_1[c_0, \overline{y}_0 - c_0] \frac{1}{1 + r_0} \frac{\theta - 1}{\theta}, \quad (30)
\]

\[
u_2[c_1(s), \overline{y}_1(s) - c_1(s)] = u_1[c_1(s), \overline{y}_1(s) - c_1(s)] \frac{1}{1 + r_1(s)} \frac{\theta - 1}{\theta}. \quad (31)
\]

If the initial prices, \( p_0(j) \), are freely chosen, then the supply condition (30) determines \( c_0 \). Given \( c_0 \), the demands for the second-period commodities are given by the Euler equations (28):

\[
u_1[c_1(s), \overline{y}_1(s) - c_1(s)] = \frac{1}{\beta f(s)} \frac{P_1(s)}{P_0} \frac{\mu(s)}{1 + r_0} u_1[c_0, \overline{y}_0 - c_0]. \quad (32)
\]

Assuming that the second-period prices, \( P_1(s) \), are exogenously given is equivalent to removing the second-period supply conditions (31) and letting the equilibrium output determined solely by the demand equations (32). Therefore, the degree of indeterminacy is unchanged. The fact that different prices are associated with different allocations is clearly seen in the demand equations (32). The reason that setting prices in advance does not change the degree of indeterminacy is that same. In such an economy, the initial-period supply condition
is absent and the second-period supply conditions (31) are replaced by a single equation (29).

As in the previous section, considering money-supply policy instead of interest-rate policy does not change the degree of indeterminacy. In the $T$-period economy, $P_t$ and $\mu$ are not determined. The degree of indeterminacy is therefore equal to the number of terminal nodes, just as in the two-period economy.

4 Staggered Prices Setting

To see that the above results are robust regarding the form of price stickiness, we consider another version of sticky prices: prices set in a staggered manner.

Suppose that at the beginning of the initial period each household is allocated into one of two groups. Households in the first group must set the first-period price of its product, $p_0(j)$, at $\bar{p}$, but they can charge the second-period price, $p_1(s,j)$, freely. Households in the second group, on the other hand, can charge the first-period price freely, but they must charge the same price in the second period, thus $p_0(j) = p_1(s,j)$, all $s \in S$. The allocation of households into these two groups is done stochastically, and the probability that each household is allocated to each group is $1/2$. We assume that there is perfect risk sharing among households.

A further restriction on the flow utility function simplifies the argument.

Assumption 4. The flow utility function is additively separable: $u(c) + v(\bar{y} - y)$, with the property that

$$\lim_{c \to 0} cu'(c) > 0.$$
where \( c_0 \) and \( c_1(s) \) denote aggregate consumption.

Since there is perfect risk sharing among households, consumption is identical between the two groups:

\[
c_0^1 = c_0^2 = c_0, \quad \text{and} \quad c_1^1(s) = c_1^2(s) = c_1(s), \quad s \in S.
\]

The first-order conditions with respect to \( c_0 \) and \( c_1(s) \) lead to

\[
\frac{\beta u'[c_1(s)]f(s)}{u'[c_0]} = \frac{P_1(s)}{P_0} \frac{\mu(s)}{1 + r_0}.
\]

The first-order condition with respect to the second-period price charged by the household in the two groups:

\[
\frac{\beta u'[c_1(s)]f(s)}{u'[c_0]} = \frac{P_1(s)}{P_0} \frac{\mu(s)}{1 + r_0}.
\]

The first-order condition with respect to the price charged in both periods by the second group of households, \( p_1(s, j) \), is given as: for each \( s \in S \),

\[
v'[\bar{y}_1(s) - y_1^1(s, j)] = u'[c_1(s)] \frac{p_1(s, j)}{P_1(s)} \frac{\theta - 1}{\theta} \frac{1}{1 + r_1(s)}.
\]

The first-order condition with respect to the price charged in both periods by the second group of households, \( p_0(j) \), is

\[
y_0^2(j) \left( v'[\bar{y}_0 - y_0^2] - u'[c_0] \frac{p_0(j)}{P_0} \frac{\theta - 1}{\theta} \frac{1}{1 + r_0} \right) + \beta \sum_s y_1^2(s, j) \left( v'[\bar{y}_1(s) - y_1^2(s)] - u'[c_1(s)] \frac{p_0(j)}{P_1(s)} \frac{\theta - 1}{\theta} \frac{1}{1 + r_1(s)} \right) f(s) = 0.
\]

In a symmetric equilibrium, households in the same group choose the same prices, so that we can write

\[
p_0(j) = p_0, \quad p_1(s, j) = p_1(s), \quad y_0^1(j) = y_0^2(j), \quad y_1^1(s, j) = y_1^2(s, j).
\]

By definition, the price levels, \( P_0 \) and \( P_1(s) \), are given by

\[
P_0 = \left[ \frac{1}{2} p_0^{1-\theta} + \frac{1}{2} p_0^{1-\theta} \right]^{-\frac{1}{1-\theta}},
\]

\[
P_1(s) = \left[ \frac{1}{2} p_1(s)^{1-\theta} + \frac{1}{2} p_1(s)^{1-\theta} \right]^{-\frac{1}{1-\theta}}.
\]

Note that \( P_1(s)/P_0 \) is an increasing function of \( p_1(s)/P_1(s) \), and \( p_0/P_1(s) \) is a decreasing function of \( p_1(s)/P_1(s) \). Production of differentiated products is given by

\[
y_0^1 = \left( \frac{p}{P_0} \right)^{-\theta} c_0, \quad y_1^1(s) = \left( \frac{p_1(s)}{P_1(s)} \right)^{-\theta} c_1(s),
\]

\[
y_0^2 = \left( \frac{p_0}{P_0} \right)^{-\theta} c_0, \quad y_1^2(s, j) = \left( \frac{p_0}{P_1(s)} \right)^{-\theta} c_1(s).
\]

Consider interest-rate policy, \( \{r_0, r_1(s), s \in S\} \). The next proposition shows that this economy, once again, has \( S \)-dimensional indeterminacy, indexed by the initial price \( P_0 \) and the nominal equivalent martingale measure \( \mu \).
Proposition 5. If second-period prices are set in a staggered manner, and if Assumptions 1 and 4 are satisfied, then, given initial nominal wealth, \( w_0 = W_0 \), and price level for the constrained households, \( \bar{p} \), interest-rate policy, \( \{ r_0, r_1(s), s \in S \} \), and fiscal policy, \( \{ \alpha, \bar{W}_1(s) \} \),

1. a competitive equilibrium exists;

2. the initial price level, \( P_0 \), and the nominal equivalent martingale measure, \( \mu \), are indeterminate;

3. the indeterminacy is real: different \( P_0 \) or different \( \mu \) are associated with different allocations.

Proof Let \( P_0 \) and \( \mu \) be given. Note that \( P_0 \) determines \( p_0 \) by (36). Consider the first-order conditions (34):

\[
v'(\bar{y}_1(s) - \left( \frac{p_1(s)}{P_1(s)} \right)^{-\theta}) c_1(s) = u'[c_1(s)] \left( \frac{p_1(s)}{P_1(s)} \right)^{\frac{\theta - 1}{\theta}} \frac{1}{1 + r_1(s)},
\]

for each \( s \in S \). Given our assumption, these equations imply that \( c_1(s) \) is a strictly increasing function of \( p_1(s)/P_1(s) \). Given the fact that \( P_1(s)/P_0 \) is a strictly increasing function of \( c_0 \), the first-order conditions (33) determines \( p_1(s)/P_1(s) \), as a strictly increasing function of \( c_0 \). Then, consider (35), and note that the left-hand side of this equation is strictly increasing in \( c_0 \) and becomes negative as \( c_0 \to 0 \). Hence, there is a unique \( c_0 \). Note that \( c_1(s), p_1(s)/P_1(s) \), and \( P_1(s)/P_0 \) are functions of \( c_0 \) and derived above. This completes the proof. \( \square \)

The intuition why the degree of indeterminacy remains the same is the same: fixing a certain number of prices removes the same number of supply conditions.

5 Infinite Horizon

It is straightforward to see that the results in the previous sections extend to the infinite-horizon economies. A general argument would be given as follows. Consider an infinite-horizon economy in which a shock, \( s_t \in S \), realizes at the beginning of each date \( t = 0, 1, 2, \ldots \). Now fix the initial price level \( P_0 \) and the nominal equivalent martingale measure \( \mu \) (if the prices are set in advance, fix \( P_1 \)). Consider a sequence of \( T \)-period truncated economies. The nominal equivalent martingale measure \( \mu \) for the infinite-horizon economy defines in an obvious way the nominal equivalent martingale measure for each truncated economy, \( \mu^T \). Given \( P_0 \) (\( P_1 \) when prices are set in advance) and \( \mu^T \), the equilibrium exists for each truncated economy. The pointwise limit of the sequence of the equilibria in the truncated economies is an equilibrium for the infinite-horizon economy. It follows that in
an infinite-horizon economy, the initial price level, $P_0$ ($P_1$ when prices are set in advance), and the nominal martingale measure, $\mu$, are indeterminate.

Given that most macro models are set in infinite horizon, however, we provide more detailed analysis on the infinite-horizon economy. Suppose that shocks follow a Markov chain with transition probabilities $f(s'|s) > 0$. The history of shocks up through date $t$ is denoted by $s^t = (s_0, \ldots, s_t)$, and called a date-event. The initial shock, $s_0$, is given, and the initial date-event is denoted by 0. The probability of date-event $s^t$ is $f(s^t)$. Successors of date-event $s^t$ is $s^{t+j}|s^t$. For $s^{t+j}|s^t$, the probability that $s^{t+j}$ occurs, conditional on $s^t$, is $f(s^{t+j}|s^t)$.

Let $\mu$ denote the nominal equivalent martingale measure. It is a probability measure over the date-event tree with $\mu(s^t) > 0$, all $s^t$. Let $q(s^{t+1}|s^t)$ denote the price at $s^t$ of the elementary security that pays off one unit of currency if and only if the date-event $s^{t+1}$ is reached. Then,

$$q(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)},$$

where $r(s^t)$ is the nominal interest rate at $s^t$. More generally,

$$q(s^j|s^t) = \frac{1}{1 + r(s^j)} \cdots \frac{1}{1 + r(s^{j-1})} \mu(s^j|s^t),$$

where $q(s^j|s^t)$ is the price at $s^t$ of the contingent claim that pays off one unit of currency if and only if $s^j$ is reached ($q(s^j|s^t) = 1$).

For simplicity, consider the flexible-price economy. The representative household has preference given by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c(s^t), \bar{y}(s_t) - y(s^t)] f(s^t),$$

where $c(s^t)$ is consumption at $s^t$; $\bar{y}(s_t)$ is the endowment; $y(s^t)$ is production.$^{15}$ At each date-event, the asset market opens first, followed by the goods market. As in Section 2, the constraints the household faces are summarized by: (i) the flow budget constraints:

$$P(s^t)c(s^t) + \frac{r(s^t)}{1 + r(s^t)} m(s^t) + \frac{1}{1 + r(s^t)} \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) w(s^{t+1}) \leq w(s^t) + \tau(s^t) + P(s^1) y(s^t),$$

(ii) the cash constraints:

$$m(s^t) \geq P(s^t) y(s^t),$$

and (iii) the natural debt limit (Lungqvist and Sargent, 2000):

$$w(s^t) \geq -\sum_{j=1}^{\infty} \sum_{s^j|s^t} q(s^j|s^t) [P(s^j) \bar{y}(s_j) + \tau(s^j)].$$

$^{15}$We have assumed that the endowment, $\bar{y}$, depends only on the current shock.
Here, $P(s^t)$ is the price level at $s^t$; $m(s^t)$ is the nominal balances carried over from $s^t$ to the next period; $w(s^{t+1})$ is the nominal value of the financial asset at the beginning of $s^{t+1}$; $r(s^t)$ is the nominal transfer from the fiscal authority at $s^t$. The natural debt limit (43) is, of course, equivalent to

$$\lim_{j\to\infty} \sum_{s^t|s^j} q(s^j|s^t)w(s^j) \geq 0.$$  

(44)

Remember that when the goods market follow the asset market, the cash-in-advance constraint is equivalent to (42).

The first-order conditions are given by

$$u_1[c(s^t), y(s^t)] = 1 + r(s^t),$$  

(45)

and the transversality condition is

$$\lim_{j\to\infty} \sum_{s^t|s^j} q(s^j|s^t)w(s^j) = 0.$$  

(47)

The flow budget constraint of the monetary-fiscal authority is

$$\frac{r(s^t)}{1 + r(s^t)} M(s^t) + \frac{1}{1 + r(s^t)} \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) W(s^{t+1}) = W(s^t) + T(s^t),$$  

(48)

where $W(0) = w(0)$ is given. Monetary policy sets either a path of nominal interest rates, \{r(s^t)\}, or a path of money supplies, \{M(s^t)\}. Fiscal policy sets a path of transfers, \{T(s^t)\}, and a path of liabilities, \{W(s^t)\}, that satisfy the flow budget constraint. In the infinite-horizon economy, fiscal policy is said to be Ricardian if it guarantees that the present discounted value of the public liability converges to zero:

$$\lim_{j\to\infty} \sum_{s^t|s^j} q(s^j|s^t)W(s^j) = 0,$$  

(49)

for any path of $q$, $P$, $r$, $M$, in or out of equilibrium. Of course, (49) is satisfied at equilibrium because of the transversality condition (47) of the household. As in the two-period case, let \{W(s^{t+1})\}_{s^{t+1}|s^t} denote the composition of the debt portfolio and $d(s^t)$ the scale of the debt. Let $H(s^t) \equiv T(s^t)/P(s^t)$ denote the real amount of the transfer from the fiscal authority. We consider the following specific forms of fiscal policy.

**Ricardian policy** Given the initial liability, $W(0)$, the fiscal authority sets sequences \{a(s^t)\}, $\alpha \leq a(s^t) \leq 1$, all $s^t$, with $0 < \alpha < 1$, and \{W(s^{t+1})\}, $W(s^{t+1}) \neq 0$. At each date-event, the nominal transfer, $T(s^t)$, and the scale of the debt, $d(s^t)$, are determined by the flow budget constraint:

$$T(s^t) = \frac{r(s^t)}{1 + r(s^t)} M(s^t) - a(s^t)W(s^t),$$

22
\[ d(s') \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)} \overline{W}(s^{t+1}) = [1 - \alpha(s^t)]W(s'). \]

**Non-Ricardian policy**  Given \( W(0) > 0 \), the fiscal authority sets a path of real transfers, \( H(s^t), \overline{W}(s^{t+1}) > 0 \). At each date-event, the scale of debt, \( d(s^t) \), is determined by the flow budget constraint:

\[ d(s^t) \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)} \overline{W}(s^{t+1}) = W(s^t) - \frac{r(s^t)}{1 + r(s^t)} M(s^t) + P(s^t)H(s^t). \]

In a deterministic setup, Woodford (1994) and Benhabib, Schmitt-Grohé and Uribe (2001) considered the non-Ricardian fiscal policy \( H_t = H \). In a stochastic setup, Dubey and Geanakoplos (2000) considered the non-Ricardian policy \( H(s^t) = 0 \); Bloise, Dréze and Polemarchakis (2002) and Dréze and Polemarchakis (2000) considered the Ricardian fiscal policy \( \alpha(s^t) = 1 \).

Note, however, that the following form of fiscal policy does not fit into any of the two regimes considered here:

\[ M(s^t) = M(s^{t-1}) + T(s^t). \]

This form of fiscal policy is considered, among others, in Woodford (1994). This is not Ricardian because it may violate (49) when the nominal interest rate becomes very low. To see this, suppose that the money supply grows at a constant rate, \( g \):

\[ M(s^t) = (1 + g)^{t+1} \overline{M}, \overline{M} > 0. \]

Since \( W(s^t) = M(s^{t-1}) \) with such fiscal policy,

\[ \sum_{s^t} q(s^t|0)W(s^t) = \sum_{s^t} q(s^t|0)M(s^{t-1}) \]

\[ = \left( \sum_{s^t} q(s^t|0) \right) (1 + g)^t \overline{M} \]

\[ = \left( \sum_{s^t} \frac{1}{1 + r(0)} \cdots \frac{1}{1 + r(s^{t-1})} \mu(s^t|0) \right) (1 + g)^t \overline{M}. \]

This does not converge to zero if the nominal interest rate becomes less than \( g \) almost surely. This is why “deflationary equilibria” are ruled out when \( g \geq 0 \) in Woodford (1994).

### 5.1 Equilibria with interest rate policy

Consider an interest rate policy, \( \{r(s^t)\} \). The flow utility function, \( u(c, l) \), satisfies Assumptions 1-2. A competitive equilibrium is defined as in the two-period economy.

With Ricardian fiscal policy, \( \{\alpha(s^t), \overline{W}(s^{t+1})\} \), there exists a continuum of equilibria indexed by the initial price, \( P(0) \), and the nominal equivalent martingale measure, \( \mu \), and the indeterminacy is nominal. To see this, pick \( P(0) \) and \( \mu \) arbitrarily. With interest
rates, \{r(s^t)\}, the first-order condition (45), together with the market clearing condition, \(c(s^t) = y(s^t)\), determines the allocation, \(\{c(s^t), y(s^t)\}\):

\[
\frac{u_1[c(s^t), \beta(s_t) - c(s^t)]}{u_2[c(s^t), \beta(s_t) - c(s^t)]} = 1 + r(s^t).
\]

Given \(P(0)\), price levels, \(P(s^{t+1})\), \(t \geq 0\), are determined by the first-order condition (46):

\[
\frac{P(s^{t+1})}{P(s^t)} = \frac{\beta u_1[c(s^{t+1}), \beta(s_{t+1}) - c(s^{t+1})]f(s^{t+1}|s^t)P(s^{t+1})}{u_1[c(s^t), \beta(s_t) - c(s^t)]} \left(1 + r(s^t)ight)
\]

The transversality condition (47) is guaranteed to hold by the fiscal policy.

On the other hand, with the non-Ricardian policy considered here, there exists a unique equilibrium. This is because there is only a unique path of price levels that is consistent with the transversality condition (47). Specifically, the unique equilibrium is constructed as follows. As in the case of Ricardian policy, interest rate policy determines the equilibrium allocation, \(\{c(s^t), y(s^t)\}\), uniquely. The individual optimization implies that

\[
\frac{W(0)}{P(0)} = \sum_{t=0}^{\infty} \sum_{s^t} q(s^t|0)P(s^t) \left\{ \frac{r(s^t)}{1 + r(s^t)} \frac{M(s^t)}{P(s^t)} - H(s^t) \right\}
\]

Since the equilibrium allocation is unique, the right hand side of the second equation is unique. Since \(W(0) > 0\), there is a unique \(P(0) > 0\) that solves the equation. Given \(P(0), q(s^t|0)P(s^t)\) are determined uniquely by

\[
q(s^t|0)P(s^t) = \frac{\beta u_1[c(s^t), \beta(s_t) - y(s^t)]f(s^t)}{u_1[c(0), \beta(0) - y(0)]}.
\]

The paths of debt and prices, \(\{W(s^{t+1}), P(s^t), q(s^t|0)\}\) are, then, determined inductively as follows. Suppose that \(P(s^t), q(s^t|0), P(s^{t+1})\) have been determined. At each date-event \(s^{t+1}\) we have

\[
W(s^{t+1}) = \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|s^{t+1})P(s^{t+j}) \left\{ \frac{r(s^{t+j})}{1 + r(s^{t+j})} c(s^{t+j}) - H(s^{t+j}) \right\}.
\]

Using \(W(s^{t+1}) = d(s^t)W(s^{t+1})\),

\[
d(s^t) = \frac{1}{W(s^{t+1})} \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|s^{t+1})P(s^{t+j}) \left\{ \frac{r(s^{t+j})c(s^{t+j})}{1 + r(s^{t+j})} - H(s^{t+j}) \right\}.
\]

Taking the summation over \(s^{t+1}|s^t\), and rearranging terms,

\[
d(s^t) = \frac{1 + r(s^t)}{q(s^t|0)} \sum_{s^{t+1}|s^t} \frac{1}{W(s^{t+1})} \times \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|0)P(s^{t+j}) \left\{ \frac{r(s^{t+j})c(s^{t+j})}{1 + r(s^{t+j})} - H(s^{t+j}) \right\}.
\]
This determines \( d(s^t) \), and hence \( W(s^{t+1}) = d(s^t)W(s^t) \). Given \( W(s^{t+1}) \), \( P(s^{t+1}) \) are determined by (51); given \( P(s^{t+1}) \), \( q(s^{t+1} | 0) \) are determined by (50), which determine \( \mu(s^{t+1} | s^t) \). Note that, since \( W(s^t) > 0 \), \( H(s^t) < 0 \), all \( s^t \) and \( t \), \( W(s^t) > 0 \), all \( s^t \) and \( t \). Hence, the resulting \( P(s^t) \) and \( \mu(s^t) \) are all positive. The path of money supplies are given by \( M(s^t) \geq p(s^t)c(s^t) \) (equality when \( r(s^t) > 0 \)).

### 5.2 Equilibria with money supply policy

Now consider money supply policy \( \{ M(s^t) \} \). Define \( c^*(s_t) \) by

\[
  c^*(s_t) = \arg \max_c u(c, \overline{y}(s_t) - c).
\]

Assume that the flow utility function, \( u(c, l) \), satisfies Assumptions 1 and 3.

With Ricardian fiscal policy, \( \{ \alpha(s^t), \overline{W}(s^{t+1}) \} \), again, there exists a continuum of equilibria, indexed by \( P(0) \) and \( \mu \). As discussed in Woodford (1994), the indeterminacy in this case is real. To see this, choose strictly positive \( P(0) \) and \( \mu \) arbitrarily. Then, a unique equilibrium corresponding to them is constructed as follows. Given \( M(0) \) and \( P(0) \), \( c(0) \), \( y(0) \), and \( r(0) \) are determined by

\[
  c(0) = \min \left\{ \frac{M(0)}{P(0)}, c^*(0) \right\}, \quad 1 + r(0) = \frac{u_1(c(0), \overline{y}(0) - c(0))}{u_2(c(0), \overline{y}(0) - c(0))},
\]

and \( y(0) = c(0) \). The rest of the equilibrium paths of \( c \), \( y \), \( P \), and \( r \) are determined inductively. If \( c(s^t), y(s^t), P(s^t) \), and \( r(s^t) \) have been determined, for a successor date-event \( s^{t+1} | s^t \), if

\[
  M(s^{t+1}) > \frac{\beta^s c^*(s_{t+1})u_1[c^*(s_{t+1}), \overline{y}(s_{t+1}) - c^*(s_{t+1})] f(s^{t+1} | s^t)}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \mu(s^{t+1} | s^t) P(s^t),
\]

then \( c(s^{t+1}) = c^*(s_{t+1}) \); otherwise, \( c(s^{t+1}) \) is a solution to

\[
  M(s^{t+1}) = \frac{\beta^s c(s^{t+1})u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - c(s^{t+1})] f(s^{t+1} | s^t)}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \mu(s^{t+1} | s^t) P(s^t).
\]

The unique existence of a solution is guaranteed by Assumption 3. Given \( c(s^{t+1}), y(s^{t+1}) = c(s^{t+1}) \), \( P(s^{t+1}) \)

\[
  P(s^{t+1}) = \frac{\beta^s u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - c(s^{t+1})] f(s^{t+1} | s^t)}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \mu(s^{t+1} | s^t) P(s^t),
\]

and

\[
  1 + r(s^{t+1}) = \frac{u_1[c(s^{t+1}), \overline{y}(s^{t+1}) - c(s^{t+1})]}{u_2[c(s^{t+1}), \overline{y}(s^{t+1}) - c(s^{t+1})]}.
\]

The path of debt portfolio, \( \{ W(s^{t+1}) \} \), is

\[
  W(s^{t+1}) = w(s^{t+1}) = \left[ 1 - \alpha(s^t) \right] q(s^t) \frac{\overline{W}(s^{t+1})}{\sum_{\tilde{s}^{t+1}|s^t} q(\tilde{s}^{t+1}) \overline{W}(\tilde{s}^{t+1})} W(s^t).
\]
It is straightforward to see that the constructed paths satisfy the first-order conditions (45)-(46) and the transversality condition (47). Clearly, the indeterminacy is real.

Note that Assumption 3 plays a crucial role here. If \( cu_1(c, y - c) \) does not converge to zero as \( c \to 0 \), then there are no equilibria in which real balances, \( M/P \), converge to zero, as discussed in Woodford (1994).

As in the two-period economy, as long as the money supply does not converge to zero too fast, there always exists an equilibrium in which the nominal interest rate equals zero at all date-events. A sufficient condition for the existence of such equilibria is, for instance, that for all \( s', s^{t+1}|s^t \), and \( t \),

\[
\frac{M(s^{t+1})}{M(s^t)} \geq \max_{s', s''} \frac{\beta u_1[c^*(s'), \bar{y}(s') - c^*(s')]}{u_2[c^*(s), \bar{y}(s) - c^*(s)]}.
\]

### 5.3 Recursive equilibria

Lucas and Stokey (1987) obtained a unique recursive equilibrium under money supply policy. With interest rate policy, however, there is a continuum of recursive equilibria.

Suppose that fiscal policy is Ricardian and interest rate policy is \( r(s') = r(s_t) \): the one-period rate depends only on the contemporaneous shock. In this case, the allocation also depends only on the current shock: \( c(s') = c(s_t) \) and \( g(s') = g(s_t) \). Of course, the stationarity of real allocations does not imply that the equilibrium inflation rate process, \( \pi(s') = P(s')/P(s'^{-1}) \), is Markov, because \( \mu \) can be any probability measure over date-events. Let \( g(s') \) denote the growth rates of money supply: \( g(s') \equiv M(s')/M(s'^{-1}) \). A recursive equilibrium is defined as an equilibrium in which the inflation rate and the money growth rate depend only on the shocks in the current and the previous periods:

\[
\pi(s^{t+1}) = \pi(s_t, s_{t+1}), \quad \text{and} \quad g(s^{t+1}) = g(s_t, s_{t+1}).
\]

The inflation rate follows such a process if and only if the nominal equivalent martingale measure is Markovian: \( \mu(s_{t+1}|s^t) = \mu(s_{t+1}|s_t) \). Hence, a recursive equilibrium is an equilibrium of the form \( \{c(s), y(s), \pi(s, s'), g(s, s'), \mu(s'|s)\} \).

The indeterminacy of such an equilibrium is shown as follows. Let \( \mu(s, s') \) be any strictly positive Markov transition probabilities. The allocation, \( \{c(s), y(s)\} \), is determined by the interest rate policy, \( \{r(s)\} \). The equilibrium inflation and money growth processes are then given by

\[
\pi(s, s') = \beta u_1[c(s'), \bar{y}(s') - c(s')] f(s'|s) \mu(s'|s),
\]

and, if \( r(s) > 0 \), all \( s \in S \),

\[
g(s, s') = \pi(s, s') \frac{c(s)}{c(s')},
\]

for all \( s, s' \in S \).
If, in addition, we require that the money growth process depends only on the current shock, \( g(s, s') = g(s') \), all \( s, s' \in S \), then the nominal equivalent martingale measure, \( \mu(s, s') \), is uniquely determined.

### 6 The Timing of Markets

So far, we have considered the economy in which, at each date-event, the asset market opens before the goods market. The result that competitive equilibria are indexed by the initial price and the nominal equivalent martingale measure does not depend on the timing of the two markets.

Consider the flexible-price economy and suppose that the goods market opens first at each date-event. Letting \( m(s^{t-1}) \) be the cash carried over from the previous date-event, the cash-in-advance constraint is written as

\[
P(s^t)c(s^t) \leq m(s^{t-1}).
\]

(52)

The amount of cash after the transactions in the goods market is

\[
m(s^{t}) + \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)} b(s^{t+1})
\]

\[
= b(s^t) + \tau(s^t) + m(s^{t-1}) - P(s^t)c(s^t) + P(s^t)y(s^t),
\]

where \( b(s^{t+1}) \) are the amount of elementary securities purchased by the household. Letting \( w(s^t) = m(s^{t-1}) + b(s^t) \), the flow budget constraint is rewritten as (41). Given prices, \( w(0) \), and \( \bar{m} \), the household maximizes (40) subject to the flow budget constraint (41), the cash constraint (52), and the natural debt limit (43). The first-order conditions for utility maximization are (52), (41),

\[
\frac{r(s^t)}{1 + r(s^t)} = \sum_{s^{t+1}|s^t} \beta f(s^{t+1}|s^t) \frac{u_1(s^{t+1}) - u_2(s^{t+1})}{u_2(s^t)} \frac{P(s^t)}{P(s^{t+1})},
\]

(53)

and

\[
\frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)} \frac{P(s^{t+1})}{P(s^t)} = \frac{\beta u_2(s^{t+1}) f(s^{t+1}|s^t)}{u_2(s^t)},
\]

(54)

where \( u_i(s^t) = u_i[c(s^t), y(s^t) - g(s^t)] \).

We assume the following:

**Assumption 5.** The flow utility function, \( u \), has the property that for all \( y > 0 \),

\[
\lim_{c \to 0} cu_2(c, y - c) = 0.
\]

27
As in the previous models, there is indeterminacy indexed by the initial price level and the nominal equivalent martingale measure. The difference is that the indeterminacy is real even with interest rate policy.

**Proposition 6.** If the good market precedes the markets for assets, and if Assumptions 1-2 and 5 are satisfied, then, given initial nominal wealth, \( w_0 = W_0 \), and money balances, \( \bar{m} = \bar{M} > 0 \), interest rate policy, \( \{r(s^t)\} \), \( r(s^t) \geq 0 \), and Ricardian fiscal policy, \( \{\alpha(s^t), \overline{W}(s^{t+1})\} \),

1. a competitive equilibrium exists;
2. the initial price, \( P(0) \), and the nominal equivalent martingale measure, \( \mu \), are indeterminate; and
3. the indeterminacy is real.

**Proof** The first-order conditions (53)-(54) are combined as

\[
1 + r(s^t) = \sum_{s^t+1 | s^t} \frac{u_1(s^t+1)}{u_2(s^t+1)} \mu(s^t+1 | s^t). \tag{55}
\]

Combining the cash-in-advance constraint (52) and the first-order condition (54) yields

\[
m(s^t) \geq (1 + r(s^t)) \frac{\beta u_2(s^{t+1}) c(s^{t+1}) f(s^{t+1}|s^t)}{u_2(s^t)} \frac{1}{\mu(s^{t+1}|s^t)} P(s^t), \tag{56}
\]

which holds with equality if \( P(s^{t+1}) c(s^{t+1}) = m(s^t) \), and thus if \( c(s^{t+1}) < c^*(s_{t+1}) \). Now, pick \( P(0) \) and \( \mu \) arbitrarily. Then, \( c(0) \) is determined as

\[
c(0) = \min \left\{ \frac{\bar{M}}{P(0)}, c'(0) \right\},
\]

and, of course, \( y(0) = c(0) \). The rest of the equilibrium paths of \( c \), \( y \), \( m \), and \( P \) are determined inductively. Suppose that \( c(s^t) \), \( y(s^t) \), \( m(s^{t-1}) \), and \( P(s^t) \) have already been determined. Then \( m(s^t) \) and \( c(s^{t+1}) \) are computed by solving (55)-(56). Set \( y(s^{t+1}) = c(s^{t+1}) \) and

\[
P(s^{t+1}) = (1 + r(s^t)) \frac{\beta u_2(s^{t+1}) c(s^{t+1}) f(s^{t+1}|s^t)}{u_2(s^t)} \frac{1}{\mu(s^{t+1}|s^t)} P(s^t). \]

The reason why indeterminacy is real in this case is clear from the cash constraint (52). If the goods market opens first, the amount of cash used to purchase consumption goods at \( s^t \) is \( m(s^{t-1}) \), which is predetermined. Hence, different prices, \( P(s^t) \), are associated with different allocations.

With money supply policy, we could show the same: the initial price and the nominal equivalent martingale measure are indeterminate, and the indeterminacy is real.

28
The results in this section are closely related to those discussed in Carlstrom and Fuerst (2001). They considered the money-in-the-utility model without uncertainty, and compared the case in which the flow utility function takes the form of \( u(c_t, M_{t-1}/P_t) \) and the case in which it is \( u(c_t, M_t/P_t) \). The latter form of the flow utility function corresponds to the cash-in-advance economy in which the asset market opens before the goods market; the former to the one in which the goods market opens before the asset market.

References


