Abstract

This paper discusses demand behavior of consumers and existence of equilibria for the standard capital asset pricing model (CAPM) with one riskless and finitely many risky assets, mean variance preferences of consumers, and subjective expectations. By treating individual expectations explicitly and parametrically, the model encompasses the description of individual as well as aggregate demand behavior for heterogeneous expectations. The paper provides a basic factorization formula for individual asset demand which implies the mutual fund property for agents with homogeneous expectations. This approach unveils some of the hidden structural features of the CAPM model often not discussed in the literature. Applying notions from standard static consumer theory, a characterization of the demand for risk from assumptions on risk preference is provided.

The paper provides sufficient conditions on preferences to generate differentiable and globally invertible asset demand behavior of consumers parameterized by wealth and arbitrary subjective expectations. In addition, the paper proves existence, uniqueness, and determinacy of equilibria for the case of arbitrary homogeneous expectations, thus complementing, amending, and generalizing existing results. Examples indicate to what extent the conditions are necessary.

Key words: Capital asset pricing, mean variance preferences, asset demand, asset market equilibrium, existence, uniqueness, determinacy, expectations.
JEL classification: E17, G12, O16.
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1 Introduction

The capital asset pricing model (CAPM)\(^1\) has become one of the most widely used models to describe demand behavior and equilibria in asset markets, deducing such well known results as the 'mutual fund theorem' or the so called 'beta pricing' relation. Although the terminology suggests that the \textit{pricing} of assets is an integral part of the model, the results are mostly given on rates of returns rather than making asset prices fully explicit\(^2\). In addition, most presentations employ a static two period version of the model invoking the notion of a temporary asset market equilibrium within an environment of uncertainty for all agents, treating the equilibrium for given expectations which often are also not made explicit. In this case, the role of the expectations on demand and/or existence of temporary equilibria cannot be studied. Since the static model precludes a systematic analysis of the stochastic nature of the underlying random process on returns, the relationship between the expectations, demand, and realizations cannot be studied as well. Few studies analyze the full stochastic embedding into an infinite sequential economy with exogenous random perturbations\(^3\).

In an attempt to embed the CAPM into a dynamic model with overlapping generations of agents (see Böhm & Chiarella 2000), it was necessary to establish existence and uniqueness of temporary equilibria of the CAPM type model in an extended economic framework with explicit parameterizations of incomes, prices, and expectations. In so doing, a multiplicative factorization of asset demand was obtained as well as some additional structural relationship of equilibrium asset prices were derived. Altogether these results shed some new light on the CAPM model and its usefulness for a dynamic analysis of asset markets.

The model used here to describe the temporary situation of an economy is (at first) a simplified version of the standard two period model of an economy with given real wealth of agents, facing uncertainty about next periods realization of the real value of their portfolio consisting of real savings and of \(K\) assets decided on in the current period. The primary difference relative to the existing literature consists of making all parameters of the consumer decision problem i. e. prices and expectations explicit. By keeping them parametrically and conceptually separate in the formulation of the model the structural equations reveal their separate impact on individual asset demand and on equilibrium asset prices. Thus the functional relationship between expectations and equilibrium asset prices is unlocked revealing the different impact of expectations and of preferences between them. This provides an explicit formulation of efficient portfolios and of the mutual fund property for agents with homogeneous beliefs. At the same time, heterogeneous expectations as well as constraints on budget and short sales are identified as additional sources of inefficiencies.

For the proof of existence and of uniqueness of equilibria this parametric separation also proves useful presenting a version of the beta pricing relation as a function of expectation parameters as well. While the method of proof here is different from most existing

\(^1\)as initially studied by Sharpe (1964),Lintner (1965), Mossin (1966)
\(^2\)see LeRoy & Werner (2001) as one typical example.
\(^3\)Stapleton & Subrahmanyam (1978) is one of the most notable exceptions; for others see the discussion in Böhm & Chiarella (2000)
ones, the conditions and assumptions here have their counterparts in those of the literature. Since these are made on the primitives of the agents’ characteristics, their role for existence, uniqueness, and efficiency can be identified more directly than in the existing literature. Moreover, the explicit parametric treatment allows the description of features of the equilibrium manifold.

As a consequence, this approach has several advantages and special features, which make a full dynamic analysis possible, i.e. to study, prove, and exhibit the properties of rational expectations equilibria, to analyze the long run behavior or prices, portfolios, and returns. In addition the impact of different and heterogeneous expectations rules, of learning, and other behavioral generalizations can be studied directly.

In other words, the beliefs on premia \((q^a - Rp, V^a)\) determine the mix of risky assets \((V^a)^{-1} \pi^a\) of any optimal portfolio, while the level of asset demand \(h(\pi^a) = m^a(p^a)/\rho^a\) is determined by the utility function, the safe rate \(R\) and initial wealth \(Re^a\). In other words, for all \(\pi^a\) and all \(\lambda \in \mathbb{R}_+\), demand \(\varphi^a(\lambda \pi^a)\) and the vector of subjective conditional premia \((V^a)^{-1} \pi^a\) are linearly dependent. Together with concavity of the mean–variance utility function one finds that individual asset demand \(\varphi^a\) is globally invertible (see Lemma ?? in the appendix).

## 2 The Basic CAPM Model

The model describes markets for \(K \geq 1\) risky assets, a consumption commodity which serves as numeraire and as a riskless asset with interest factor \(R > 0\). Market interaction occurs among finitely many investors who do not possess storage possibilities for the consumption good directly and who attempt to maximize the expected utility of future consumption (wealth).

### Consumers/Investors

Portfolio choices on the basis of mean–variance preferences constitute a widely accepted framework to analyze decisions under risk and their associated markets, inducing a much studied class of models successfully used in financial theory. They form the basis of the classical capital asset pricing model (CAPM) the results of which serve as a fundamental guideline to understanding and evaluating the trade off between returns and risk in asset markets.

The primary importance of mean–variance preferences within the classical asset pricing theory stems from the fact that they supply a convenient structure to analyze asset demand behavior explicitly. The case with quadratic utility and normally distributed returns yields the well known standard CAPM pricing formula of Sharpe–Lintner–Mossin. In other more general situations, as is known with quadratic utility, mean–variance preferences induce globally invertible demand functions which are often solvable algebraically. In the general equilibrium context this may yield explicit functional forms of the equilibrium price map, which are needed if an explicit description of the asset price process is

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\(^5\)Sharpe (1964), Lintner (1965), and Mossin (1966)
the goal.
In this context, the typical consumer/investor is characterized by

- $w \in \mathbb{R}$ units of a numeraire commodity as his consumption/numeraire endowment
- a one period planning horizon
- risk preferences over the mean $\mu$ and the standard deviation $\sigma$ of his future consumption/wealth described by a utility function

$$U: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$$

which is increasing in the mean $\mu$ and decreasing in the standard deviation $\sigma$.
- a subjective probability distribution $\nu \in \text{Prob}(\mathbb{R}^K)$ for future \textit{cum-dividend} asset prices (gross returns) parameterized by a pair $(q^e, V) \in \mathbb{R}^K \times \mathbb{R}^{K^2}$ of expected mean cum dividend prices $q^e \in \mathbb{R}^K$ and associated covariance matrix $V \in \mathbb{R}^{K^2}$.

**Future Wealth**

Let $(x^{(1)}, \ldots, x^{(K)}, y) = (x, y) \in \mathbb{R}^K \times \mathbb{R}$ denote a portfolio of $K + 1$ assets and let $p \in \mathbb{R}^K$ denote current asset prices in units of the numeraire commodity. Then, without (short sale) constraints on the demand for all $K + 1$ assets the investor's budget constraint requires

$$w = \langle p, x \rangle + y = \sum_{k=1}^{K} p^{(k)} x^{(k)} + y,$$

where $\langle \ldots \rangle$ denotes the scalar product of any two vectors. If $q^e := p' + d \in \mathbb{R}^K$ denotes future \textit{cum-dividend} prices of the $K$ assets his future (random) consumption/wealth becomes

$$W(x, w, p, q) = R[w - \langle p, x \rangle] + \langle q, x \rangle = Rw + \langle (q - Rp), x \rangle$$

Let $\nu \in \mathbb{P}$ denote a probability measure on $\mathbb{R}^K$ describing the distribution of future cum dividend prices, characterizing the expectations subjectively held by the investor. Then, for any asset portfolio $x \in \mathbb{R}^K$ this yields the subjectively expected value of his future wealth

$$\mathbb{E}_\nu[W(x, w, p, \cdot)] = \int_{\mathbb{R}_+^K} Rw + \langle (q - Rp), x \rangle \nu(dq) = \int_{\mathbb{R}_+^K} \langle (q^e - Rp), x \rangle$$

with associated subjective variance

$$\mathbb{V}_\nu[W(x, w, p, \cdot)] = \int_{\mathbb{R}_+^K} \left( W(x, w, p, q) - \mathbb{E}_\nu[W(x, w, p, \cdot)] \right)^2 \nu(dq) = \langle x, Vx \rangle.$$  

Concerning subjective expectations\textsuperscript{6}, the following assumption will be made throughout, which implies that the consumer assumes that none of the assets is redundant.

\textsuperscript{6}see Nielsen (1988)
2.1 Efficient Portfolios

Assumption 2.1
The subjective expectations for future cum dividend prices are parameterized in

- $q^e = p^e + d^e \in \mathbb{R}^K$ the vector of expected values of future cum dividend prices and
- $V \in \mathbb{R}^{K \times K}$ the associated covariance matrix

and satisfy:

(i) $q \in \mathbb{R}^K$,

(ii) $V \in \mathbb{R}^{K^2}$ is symmetric and positive definite.

Define $\pi := q^e - Rp$ as the vector of expected risk premia of the $K$ assets. Then, one can write

$$U\left(\mathbb{E}_\nu[W(x, w, p, \cdot)], \mathbb{V}_\nu[W(x, w, p, \cdot)]^{\frac{1}{2}}\right) = U\left(Rw + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}}\right).$$

As a consequence asset demand of an investor with $\mu - \sigma$-preferences and expectations $\nu$ can be defined as

$$\varphi(w, \pi, V) := \arg\max_{x \in \mathbb{R}^K} U\left(\mathbb{E}_\nu[W(x, w, p, \cdot)], \mathbb{V}_\nu[W(x, w, p, \cdot)]^{\frac{1}{2}}\right)$$

$$= \arg\max_{x \in \mathbb{R}^K} U\left(Rw_0 + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}}\right)$$

for every $(w, \pi, V)$ satisfying assumption 2.1

2.1 Efficient Portfolios

Definition 2.1
An asset portfolio $x \in \mathbb{R}^K$ is called mean–variance–efficient (or $\mu$–$\sigma$–efficient) with respect to $(\pi, V)$, $\pi \neq 0$, if there is no $x' \neq x$ such that

$$\langle \pi, x' \rangle \geq \langle \pi, x \rangle$$

$$\langle x, Vx \rangle \geq \langle x', Vx' \rangle$$

with at least one strict inequality.

Note that this concept of efficiency is strictly subjective and agent specific, since it depends on individual subjective beliefs, implying that, typically, a particular portfolio $x$ will not be efficient for agents with heterogeneous beliefs. Let $X_{\text{eff}}(\pi, V)$ denote the set of all $\mu$–$\sigma$–efficient portfolios. When asset prices $p$ and expectations $(q^e, V)$ are given, one often writes $X_{\text{eff}}$ instead of $X_{\text{eff}}(\pi, V)$. Similarly, one simply speaks of efficiency rather than of $\mu$–$\sigma$ efficiency.

Lemma 2.1

$$X_{\text{eff}}(\pi, V) = \left\{\lambda V^{-1}\pi | \lambda \geq 0\right\}$$

(2.5)
2.1 Efficient Portfolios

Proof: Consider $\lambda > 0$ and some $x \neq \lambda V^{-1} \pi$ with $\langle \pi, x \rangle = \langle \pi, \lambda V^{-1} \pi \rangle$. It suffices to show that

$$\langle \lambda V^{-1} \pi, V \lambda V^{-1} \pi \rangle < \langle x, V x \rangle$$

One has

$$0 < \langle (x - \lambda V^{-1} \pi), V (x - \lambda V^{-1} \pi) \rangle$$

$$= \langle x, V x \rangle - \lambda \langle V^{-1} \pi, V x \rangle$$

$$= \langle x, V x \rangle - \lambda \langle \pi, x \rangle$$

$$= \langle x, V x \rangle - \lambda \langle \pi, \lambda V^{-1} \pi \rangle$$

$$= \langle x, V x \rangle - \langle \lambda V^{-1} \pi, V \lambda V^{-1} \pi \rangle$$

which implies $\langle \lambda V^{-1} \pi, V \lambda V^{-1} \pi \rangle < \langle x, V x \rangle$ QED.

Figure 2.1: Efficient Portfolios

Figure (2.1) displays the set of efficient portfolios $X_{eff}$ for the case of two assets. It consists of all tangency points of the contours of the quadratic mapping $\langle x, V x \rangle$ with the contour lines of the expected return $\langle \pi, x \rangle$. Thus, all efficient portfolios have the same mix of assets, but they differ in their scale. Larger asset bundles imply larger means and larger variances, since they are positively homogeneous functions of the scale factor $\lambda$. It is immediate that the set of efficient portfolios can also be obtained as the family of solutions of the following standard optimization problem:

$$m(\sigma, \pi, V) := \max_{x \in \mathbb{R}^n} \left\{ \langle \pi, x \rangle | \langle x, V x \rangle^{\frac{1}{2}} \leq \sigma \right\} = \langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}} \sigma$$

(2.6)
2.2 Asset Demand with Mean–Variance Preferences

Solutions to (2.6) exist whenever $V$ is positive definite. They are unique if $\pi \neq 0$ and given by

$$\psi(\sigma, \pi, V) := \arg \max_{x \in \mathbb{R}^K} \left\{ \langle \pi, x \rangle | \langle x, Vx \rangle \leq \sigma \right\} = \frac{\sigma}{\langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}} V^{-1} \pi$$

which yields

$$X_{eff} = \left\{ x' | x' = \frac{\sigma}{\langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}} V^{-1} \pi, \sigma \geq 0 \right\}$$

(2.8)

For given $(\pi, V)$, the function (2.7) defines a linear relationship between expected mean $\mu$ and the standard deviation $\sigma$ among efficient portfolios. Its graph is the so-called efficiency line and its slope $\langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}$ is the marginal or shadow return to risk.

2.2 Asset Demand with Mean–Variance Preferences

Assumption 2.2

Risk preferences are described by a continuous quasi concave utility function of mean and standard deviation $(\mu, \sigma)$

$$U : \left\{ \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \right\}_{(\mu, \sigma)} \mapsto U(\mu, \sigma)$$

which is strictly increasing in its first argument $\mu$ and strictly decreasing in its second argument $\sigma$.

In some situations it may be convenient to include preferences which are monotonic only in a weak sense, i.e. non-decreasing in its first argument $\mu$ and non-increasing in its second argument $\sigma$. The next two propositions state some fundamental properties of asset demand.

Proposition 2.1

Let assumption (2.2) be satisfied. If $V \in \mathbb{R}^{K \times K}$ is positive definite, then

$$\varphi(w, \pi, V) := \arg \max_{x \in \mathbb{R}^K} \left( u(Rw + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}}) \right) \subset X_{eff}(\pi, V).$$

(2.9)

For every $x \in \varphi(w, \pi, V)$, there exists $\lambda \geq 0$, such that:

$$x = \lambda V^{-1} \pi$$

(2.10)

$$\langle \pi, x \rangle = \lambda \langle \pi, V^{-1} \pi \rangle$$

(2.11)

$$\langle x, Vx \rangle = \lambda^2 \langle \pi, V^{-1} \pi \rangle$$

(2.12)

$$\frac{\langle \pi, x \rangle}{\langle x, Vx \rangle^{\frac{1}{2}}} = \langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}$$

(2.13)

In other words, any demand vector is colinear to the efficient vector $V^{-1} \pi$. This means that the mix of asset demand is determined by prices and expectations while the level of demand is determined by preferences and wealth. Thus, all wealth and preference
Characteristics must be embodied in the factor $\lambda$ of equation (2.10) as a function of $(w, \pi, V)$.

Proof: Property (2.10) follows immediately from the efficiency lemma (2.1). All other properties are direct implications of (2.10) QED.

If demand is unique, the properties (2.10)- (2.13) imply that the demand function $\varphi(w, \pi, V)$ satisfies the following properties hold for all $(w, \pi, V)$ :

\begin{align*}
\varphi(w, \pi, V) &= \lambda V^{-1} \pi \quad (2.14) \\
\langle \pi, \varphi(w, \pi, V) \rangle &= \lambda \langle \pi, V^{-1} \pi \rangle \quad (2.15) \\
\langle \varphi(w, \pi, V), V \varphi(w, \pi, V) \rangle &= \lambda^2 \langle \pi, V^{-1} \pi \rangle \quad (2.16) \\
\frac{\langle \pi, \varphi(w, \pi, V) \rangle}{\langle \varphi(w, \pi, V), V \varphi(w, \pi, V) \rangle^{\frac{1}{2}}} &= \langle \pi, V^{-1} \pi \rangle \quad (2.17)
\end{align*}

Figure (2.2) provides a geometric characterization of an efficient demand bundle $x^*$ for the situation with two assets. The iso-variance lines of figure (2.1) have been superimposed by the contours of the function of feasible utilities $U(Rw_0 + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}})$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{efficient_demand.png}
\caption{Figure 2.2: Efficient demand}
\end{figure}

Notice that the number $\langle \pi, V^{-1} \pi \rangle$ plays an important role in these equations. It is simultaneously equal to the mean and to the variance of the efficient portfolio $V^{-1} \pi$. Moreover, it is also the parametrically given (linear) shadow price of the efficiency solution (2.6). In other words, $\langle \pi, V^{-1} \pi \rangle$ is the marginal (shadow) return to any additional risk given prices and expectations. This is equivalent to the statement of equation (2.13) that efficient $\mu - \sigma$ combinations have a constant ratio of $\langle \pi, V^{-1} \pi \rangle$, which is a homogeneous function of degree two in prices and expectations. Combining these findings one can now derive the
precise expression of the scale of asset demand \( \lambda \) to obtain the fundamental factorization formula of asset demand. This result also confirms the so called mutual fund theorem, i.e. agents with identical beliefs hold identical mixes of assets.

**Lemma 2.2**

\[
\varphi(w, \pi, V) := \arg \max_{x \in \mathbb{R}^K} U(w + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}})
= \frac{s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}})}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi, \tag{2.18}
\]

where

\[
s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}) := \arg \max_{\sigma \geq 0} U(Rw + \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}\sigma, \sigma). \tag{2.19}
\]

Therefore, the individual level of asset demand is equal to the ratio of the indirect standard deviation and that of the standardized portfolio \( V^{-1}\pi \).

**Proof:** One verifies easily that

\[
\max_{x \in \mathbb{R}^K} U(Rw + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}}) = \max_{x \in X_{eff}} U(Rw + \langle \pi, x \rangle, \langle x, Vx \rangle^{\frac{1}{2}})
= \max_{\sigma \geq 0} U(Rw + m(\sigma, \pi, V), \sigma)
= U(Rw + m(s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}), s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}))
= \max_{\sigma \geq 0} U(Rw + \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}\sigma, \sigma)
\]

Therefore:

\[
\varphi(w, \pi, V) := \psi(s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}), \pi, V)
= \frac{s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}})}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi, \tag{2.20}
\]

QED.

### 2.3 Budget restrictions and short sale constraints

Short sales constraints may apply when \( X_{eff} \notin \mathbb{R}^K_+ \), i.e. when efficiency to buy short. In such a case the constraints act like quantity rationing constraints. It is well known that they will cause spillovers in demand between different assets. In other words, demand with a binding short sale constraint will no longer be colinear to \( V^{-1}\pi \) in general. Thus, as the geometry of Figure (2.2) suggests, demand will not be \( (\pi, V) \) efficient. As a consequence, demand an agent deviates from the efficiency line depending on individual preferences, creating a wedge between the efficiency trade off and the risk trade off. Therefore, aggregate demand in any market situation with homogeneous beliefs will not be efficient if some agent faces a binding short sales constraint. It is evident that such situations are also not constraint efficient in general\(^7\).

\(^7\)Binding short sales constraints correspond exactly to quantity constraints as treated in the rationing literature. Therefore all results concerning efficiency and constraint efficiency carry over directly to the asset demand here (see for example Böhm & Müller 1977)
A similar loss of efficiency occurs when the budget constraint of an agent becomes binding, i.e. when he is not allowed to go short on the safe asset to obtain credit. Figure ?? describes such a situation with two assets without short sales constraint, but \( p, x^* > w \). The inefficiency (i.e. \( \tilde{x} \notin X_{eff} \)) occurs when prices \( p \) and expectation \( q \) are not colinear.

\[ \text{Figure 2.3: Inefficient of Demand } \tilde{x} \]

### 2.4 Existence of Demand

The following four examples provide additional insight into possible properties of asset demand. They also indicate which additional assumptions are needed to obtain well defined demand for all prices and expectations.

#### Example 1: Linear mean variance preferences

Let the preferences of an investor

\[
U : \begin{cases} \mathbb{R} \times \mathbb{R}_+ & \rightarrow \mathbb{R} \\ (\mu, \sigma) & \mapsto U(\mu, \sigma) \end{cases}
\]

be given by the function

\[
U(\mu, \sigma) = \mu - \frac{\alpha}{2} \sigma^2, \tag{2.20}
\]

where \( \alpha \) is usually interpreted as a measure of risk aversion. From the first order conditions one calculates directly the demand as

\[
\varphi(w, \pi, V) = \frac{1}{\alpha} V^{-1} \pi = \frac{1}{\alpha} V^{-1} (q^e - Rp) \tag{2.21}
\]

Thus, asset demand is linear (homogeneous of degree one) and globally invertible in \( \pi \), and independent of initial wealth. Notice that the investor’s (indirect) expected return is from his asset demand

\[
m(w, \pi, V) := \langle \pi, \varphi(w, \pi, V) \rangle = \frac{1}{\alpha} \langle \pi, V^{-1} \pi \rangle \tag{2.22}
\]
and his demand for risk (his indirect standard deviation) is

\[ s(\langle \pi, V^{-1}\pi \rangle^\frac{1}{2}) := \langle \varphi(w, \pi, V), V\varphi(w, \pi, V) \rangle^\frac{1}{2} = \frac{1}{\alpha} \langle \pi, V^{-1}\pi \rangle^\frac{1}{2}, \quad (2.23) \]

implying that he always chooses a standard deviation equal to his risk factor \(1/\alpha\) (sometimes called the risk tolerance) times the risk of the standardized portfolio which corresponds to efficient shadow return to risk. The homogeneity of demand implies that the investor would be willing to bear unlimited risk when the ratio of expected return to risk of the standardized portfolio becomes unbounded.

**Example 2: Logarithmic mean utility**

Consider next the situation with the utility function

\[ U(\mu, \sigma) = \ln(\mu) - \frac{\alpha}{2} \sigma^2 \quad \mu > 0, \quad (2.24) \]

which yields the asset demand function

\[ \varphi(w, \pi, V) = \frac{1}{\beta} V^{-1}\pi, \quad (2.25) \]

where

\[ \beta = \frac{(\alpha Rw + \sqrt{(\alpha Rw)^2 + 4\alpha \langle \pi, V^{-1}\pi \rangle})}{\sqrt{4(\langle \pi, V^{-1}\pi \rangle)^2}} \quad (2.26) \]

\(\beta\) is the unique positive solution of the quadratic equation

\[(\beta^2 - \alpha)\sqrt{\langle \pi, V^{-1}\pi \rangle} - \alpha \beta Rw = 0.\]

One obtains as the investors expected return

\[ m(w, \pi, V) := \langle \pi, \varphi(w, \pi, V) \rangle = \frac{1}{\beta} \langle \pi, V^{-1}\pi \rangle^\frac{1}{2} \quad (2.27) \]

and his demand for risk as

\[ s(w, \pi, V) := \langle \varphi(w, \pi, V), V\varphi(w, \pi, V) \rangle^\frac{1}{2} = \frac{1}{\beta^2} \quad (2.28) \]

For \(w = 0\) the investor will always demand constant risk equal to \(1/\sqrt{\alpha}\). Moreover, for \(w > 0\), one observes that \(\beta \geq \sqrt{\alpha}\) for all \(\pi\), which implies that his demand for risk is always less than or equal to \(\sqrt{\alpha}\), regardless of wealth, prices, and expectations. As a consequence, his asset demand will be bounded uniformly for all \((w, \pi, V)\). Moreover, the demand for risk is decreasing in wealth \(w\).

**Example 3: Quasi concave utility**

Consider next the situation with the utility function

\[ U(\mu, \sigma) = \frac{\mu^\alpha}{r + \sigma^\beta}, \quad r > 0, \quad 0 < \alpha \leq 1 \leq \beta \quad \mu \geq 0 \quad (2.29) \]
which is not concave, but strictly quasi concave as long as \( \alpha < \beta \). For \( w = 0 \), this yields the asset demand function

\[
\varphi(0, \pi, V) = \frac{1}{\langle \pi, V^{-1} \rangle^{\frac{1}{2}}} \left( \frac{r \alpha}{\beta - \alpha} \right)^{\frac{1}{2}} V^{-1} \pi
\]  

(2.30)

Notice again that the investor’s demand for risk

\[
s(0, \langle \pi, V^{-1} \rangle^{\frac{1}{2}}) = \left( \frac{r \alpha}{\beta - \alpha} \right)^{\frac{1}{2}}
\]

is constant, i.e. independent of \( w \) and \( \langle \pi, V^{-1} \rangle^{\frac{1}{2}} \). Therefore, asset demand is uniformly bounded for all \( (\pi, V) \) for \( \alpha < \beta \).

If \( \alpha = \beta = 1 \), the indifference curves of \( U \) are rotating lines with higher slopes for higher utility. One finds

\[
\varphi(R, \pi, V) = \begin{cases} 
0, & \text{if } \langle \pi, V^{-1} \rangle^{\frac{1}{2}} > \frac{Rw}{r}; \\
X_{eff}, & \text{if } \langle \pi, V^{-1} \rangle^{\frac{1}{2}} = \frac{Rw}{r}; \\
0, & \text{if } \langle \pi, V^{-1} \rangle^{\frac{1}{2}} < \frac{Rw}{r}.
\end{cases}
\]

(2.31)

Thus, demand is empty if the return to risk \( \langle \pi, V^{-1} \rangle^{\frac{1}{2}} \) is larger than \( Rw/r \). If it exists it is either zero or the whole set of efficient asset bundles \( X_{eff} \).

**Example 4: Unbounded Demand**

Finally, consider the situation with the concave utility function

\[
U(\mu, \sigma) = \mu - \frac{\sigma^2}{r + \sigma}, \quad r > 0,
\]

(2.32)

which is also strictly quasi concave. It will be shown that the demand for risk and the demand are not defined for a large range of prices and expectations.

Given \( (\pi, V) \), consider an arbitrary \( \lambda > 0 \) and an associated efficient bundle \( x(\lambda) = \lambda V^{-1} \pi \). This induces a feasible utility

\[
U(\lambda) := Rw + \lambda \langle \pi, V^{-1} \rangle - \frac{\lambda^2 \langle \pi, V^{-1} \rangle}{r + \lambda \langle \pi, V^{-1} \rangle^{\frac{1}{2}}} \\
> Rw + \lambda \langle \pi, V^{-1} \rangle^{\frac{1}{2}} \left( \langle \pi, V^{-1} \rangle^{\frac{1}{2}} - 1 \right).
\]

(2.33)

(2.34)

Thus, \( U(\lambda) \) is strictly increasing in \( \lambda \) for \( \langle \pi, V^{-1} \rangle^{\frac{1}{2}} \geq 1 \). Therefore, asset demand does not exist. For \( \langle \pi, V^{-1} \rangle^{\frac{1}{2}} < 1 \), one obtains

\[
\varphi(w, \pi, V) = \frac{r \left( 1 + \sqrt{\left( 1 - \langle \pi, V^{-1} \rangle^{\frac{1}{2}} \right)^{-1}} \right)}{\langle \pi, V^{-1} \rangle^{\frac{1}{2}}} V^{-1} \pi
\]

(2.35)
One observes that the demand for risk and therefore asset demand becomes unbounded as \( <\pi, V^{-1}\pi>^\frac{1}{2} \) approaches 1 from below.

The last two examples combined with the factorization lemma (2.2) show that properties of the utility function/the preferences and not expectations are responsible for the failure of demand to exist, since asset demand becomes unbounded if and only if the demand for risk becomes unbounded for some finite shadow return for risk. Thus, the set of convex preferences has to be restricted if well defined demand for all prices and expectations is required. This leads in a natural way to the next lemma.

**Definition 2.2** Let \( S \) denote a subset of \( \mathbb{R}^m \). The asymptotic cone of \( S \), denoted \( \mathbb{A}S \) is defined as

\[
\mathbb{A}S := \bigcap_{k=1}^{\infty} C(S^k)
\]

(2.36)

where \( C(S^k) \) denotes the smallest closed cone containing \( S^k := \{x \in S | |x| \geq k\} \).

**Lemma 2.3**

Let assumption 2.2 be satisfied and define

\[
P(\bar{\mu}, \bar{\sigma}) := \{(\mu, \sigma)| U(\mu, \sigma) \geq U(\bar{\mu}, \bar{\sigma})\}
\]

as the upper contour set for the utility function \( U(\mu, \sigma) \) at the point of \( (\bar{\mu}, \bar{\sigma}) \). If

\[
\mathbb{A}P(Rw, 0) = \{\lambda (0, 1) | \lambda \geq 0\}
\]

(2.37)

then the demand for risk \( s(w, <\pi, V^{-1}\pi>^\frac{1}{2}) \) is finite for all prices and expectations satisfying assumption (2.1).

In other words, if the asymptotic cone of the better set contains no strictly positive vectors of \( \mathbb{R}^2^+ \), i.e. it consists of the half line through \((0, 1)\) alone, demand is nonempty\(^8\).

**Proof:**

Monotonicity of the utility function allows one to rewrite the demand for risk \( s(Rw_0, <\pi, V^{-1}\pi>^\frac{1}{2}) \) as the solution of

\[
s(w, <\pi, V^{-1}\pi>^\frac{1}{2}) := \arg \max_{\sigma \geq 0} u(Rw_0 + \mu, \sigma | \mu \leq <\pi, V^{-1}\pi>^\frac{1}{2}\sigma, \sigma \geq 0).
\]

(2.38)

Observe that any demand pair \((\mu, \sigma) = (<\pi, V^{-1}\pi>^\frac{1}{2}s(w, <\pi, V^{-1}\pi>^\frac{1}{2}), s(w, <\pi, V^{-1}\pi>^\frac{1}{2}))\) must lie in \( P(Rw, 0) \cap \{(\mu, \sigma) | \mu \leq <\pi, V^{-1}\pi>^\frac{1}{2}\sigma, \sigma \geq 0\} \), which is the intersection of two closed convex sets. It is compact if the intersection of their asymptotic cones is \( \{0\} \).

Since, \( \mathbb{A}P(Rw, 0) = \{\lambda (0, 1) | \lambda \geq 0\} \) it follows that

\[
\mathbb{A}P(Rw_0, 0) \cap \mathbb{A} \{(\mu, \sigma) | \mu \leq <\pi, V^{-1}\pi>^\frac{1}{2}\sigma, \sigma \geq 0\} = \{0\}
\]

(2.39)

\[
= \{\lambda(0, 1) | \lambda \geq 0\} \cap \{(\mu, \sigma) | \mu \leq <\pi, V^{-1}\pi>^\frac{1}{2}\sigma, \sigma \geq 0\} = \{0\}
\]

(2.40)

\(^8\)Note the relationship to the discussion of useful trade vectors by Werner (1987)

\(^9\)for details see Debreu (1959) or Hildenbrand (1974)
2.4 Existence of Demand

Therefore, continuity of the utility function $U$ implies the existence of demand. QED.

It follows from results of standard consumer theory, that strict convexity of preferences implies that demand is a continuous function. However, one additional property is required to obtain non zero demand, in other words a condition is needed which induces non zero demand for risk for all positive shadow returns $\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}} > 0$.

Consider the utility contour associated with the initial wealth $R_w$

$$\{(\mu, \sigma) \mid U((\mu, \sigma)) = U(R_w, 0)\}.$$

Since $U$ is strictly increasing in $\mu$, there exists a function $h : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, $(R_w, \sigma) \mapsto h(R_w, \sigma)$, describing the indifference curve through $(R_w, 0)$. $h$ is continuous, monotonically increasing, and convex in $\sigma$. Moreover, $h(R_w, 0) = R_w$ for all $R_w$. It is evident that the curvature of the function $h$ is related to the properties of the asymptotic cone of $P(R_w, 0)$. More specifically, $\mathbb{A}P(R_w, 0)$ is equal to the half line $\{\lambda (0, 1) \} \subset \mathbb{R}^2$ if and only if

$$\lim_{\sigma \to \infty} \frac{h(R_w, \sigma) - R_w}{\sigma} = \infty.$$

Exploiting the curvature of $h$ near zero, it is evident that demand for risk will be positive for all $\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}} > 0$ if and only if

$$\lim_{\sigma \to 0} \frac{h(R_w, \sigma) - R_w}{\sigma} = 0.$$

These two properties are a direct translation of the so called weak Inada condition used in growth models to convex functions. Combining these observations with the results from lemmas (2.1), (2.2), and (2.3), one obtains the following theorem on existence, uniqueness, and positivity of demand under mean-variance preferences.

**Theorem 2.1**

Let expectations satisfy (2.1) and assume that preferences are strictly quasi concave satisfying assumption (2.2). If, in addition, the Inada conditions

$$\begin{align*}
(a) \quad & \lim_{\sigma \to 0} \frac{h(R_w, \sigma) - R_w}{\sigma} = 0 \quad \text{and} \\
(b) \quad & \lim_{\sigma \to \infty} \frac{h(R_w, \sigma) - R_w}{\sigma} = \infty,
\end{align*}\tag{2.41}$$

hold, then, for all $\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}} > 0$,

(i) asset demand is a continuous function for all $\pi \neq 0$

(ii) the demand for risk $s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}) > 0$

(iii) asset demand $\varphi(w, \pi, V) = \frac{s(w, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}})}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi \neq 0$

**Proof:**

Part (b) of the Inada conditions (2.41) together with strict quasi concavity implies existence, uniqueness, and continuity of demand. Positive demand for risk follows from part (a) of the Inada condition, which implies a non zero asset demand. QED.
2.5 Differentiable Demand

It is now straightforward to derive conditions under which one obtains differentiable demand functions by applying results from standard consumer theory. Due to the factorization lemma (2.2) it is sufficient to derive a differentiable demand function for risk. For given expectations and prices define \( \rho := \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}} > 0 \) as the shadow return (or equivalently as the standard deviation of the standardized portfolio \( V^{-1}\pi \)). Then, the demand for risk is given by

\[
s(w, \rho) := \arg \max \{ U(\mu, \sigma) \mid \mu \leq \rho \sigma + Rw \}.
\]

The demand for risk induces a demand for expected return as

\[
m(w, \rho) := Rw + \rho s(w, \rho)
\]

This maximization corresponds to a standard consumer problem with a linear budget set, where \( \rho \) is the price/return for risk taking. Since the utility is decreasing in risk equation (2.42) corresponds to a consumer’s supply of a factor to the market (like labor with real wage \( \rho \)). Nevertheless, we will continue to speak of demand for risk rather than supply. As a consequence, all further results on individual demand for risk have their complete analogues in consumer theory with endogenous labor supply allowing a direct application of all known results.

To obtain a differentiable demand function for risk the following additional assumption is necessary and sufficient (cf. for example Böhm & Barten (1982), Debreu (1972), Katzner (1968)).

**Assumption 2.3**

Let the utility function

\[
U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}
\]

\[
(\mu, \sigma) \mapsto U(\mu, \sigma)
\]

be strictly quasi concave, twice continuously differentiable and satisfy (2.2) as well as

\[
\begin{vmatrix}
U_{11} & U_{12} & U_1 \\
U_{21} & U_{22} & -U_2 \\
U_1 & -U_2 & 0
\end{vmatrix} \neq 0
\]

for all interior \( (\mu, \sigma) \).

The condition (2.44) of a non vanishing bordered Hessian matrix implies that the Gaussian curvature of all indifference curves is different from zero at all interior points, which in turn implies the differentiability of the demand function for risk \( s : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \), \( (w, \rho) \mapsto s(w, \rho) \), (see Katzner (1968) and Debreu (1982)).

**Lemma 2.4**

Given the assumptions to guarantee differentiable demand for risk and return, the follow-
Differentiable Demand

Properties hold for all interior \((w, \rho)\):

\[
\frac{\partial m}{\partial w}(w, \rho) = R + \rho \frac{\partial s}{\partial w}(w, \rho) \tag{2.45}
\]

\[
\frac{\partial m}{\partial \rho}(w, \rho) = s(w, \rho) \left( 1 + \frac{\rho}{s(w, \rho)} \frac{\partial s}{\partial \rho}(w, \rho) \right) \tag{2.46}
\]

\[
\frac{\partial s}{\partial \rho}(w, \rho) > s(w, \rho) \frac{\partial s}{\partial w}(w, \rho) \tag{2.47}
\]

**Proof:**
The first two equations are identities induced by monotonicity and the budget equation. Equation (2.47) follows from standard duality and the Slutsky equation. QED.

Combining (2.45) – (2.47) one obtains

\[
E_{\sigma,\rho}(w, \rho) > \rho \frac{\partial s}{\partial w}(w, \rho) = \frac{\partial m}{\partial w}(w, \rho) - R > -R \tag{2.48}
\]

if \(\partial m/\partial w > 0\) where

\[
E_{\sigma,\rho}(w, \rho) := \frac{\rho}{s(w, \rho)} \frac{\partial s}{\partial \rho}(w, \rho)
\]

defines the elasticity of the demand for risk with respect to \(\rho\), which is bounded below.

If wealth has a positive effect on the chosen amount of risk, \(\partial s/\partial w > 0\), this fact expresses a higher desire for risk with higher wealth. Using traditional terminology in this situation, define risk to be a normal good\(^{10}\) if \(\partial s/\partial w \geq 0\). By analogy from standard microeconomic terminology, define risk as an inferior good if \(\partial s/\partial w < 0\).\(^{11}\) Notice that what is interpreted here as demand for risk is essentially the supply of a willingness to accept risk, which corresponds to the microeconomic analogue of a household’s labor supply. Notice, that with this terminology, risk normality corresponds to the situation of labor supply increasing in wealth implying that leisure is an inferior good (a situation usually not considered as typical – or normal!)

The Slutsky equation (2.47) and risk normality imply that a higher shadow return to risk implies a higher chosen amount of risk, i.e. \(\partial s/\partial \rho > 0\). Therefore, demand for risk is globally increasing in the shadow return, a property which is necessary for unique equilibria (see Section ?? below). Clearly, risk inferiority is only a necessary condition for a decreasing demand for risk in the shadow return \(\rho\), allowing for positive as well as negative price effects. Moreover, risk normality may not be the most widely accepted economic assumption to be used for the CAPM model. Therefore, it seems desirable to elucidate those assumption under risk inferiority for which demand for risk is still globally increasing. It is difficult to obtain intuitive economic assumptions for general mean variance preferences which imply monotonicity of demand for risk in general. However, for separable utility functions one obtains the following result.

\(^{10}\)Hens, Laitenberger & Löffler (2002) call this property decreasing (non increasing) risk aversion.

\(^{11}\)called increasing risk aversion by Hens, Laitenberger & Löffler (2002)
2.5 Differentiable Demand

Theorem 2.2

Consider a separable concave twice continuously differentiable utility function

\[ U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \]

of the form

\[ U(\mu, \sigma) = u(\mu) - v(\sigma) \]  \hfill (2.49)

satisfying (2.2) and

\[ v'(0) = 0, \quad \lim_{\sigma \to -\infty} v'(\sigma) = \infty, \quad \text{and} \quad v''(\sigma) > 0 \quad \text{for all} \quad \sigma > 0. \]  \hfill (2.50)

If for all \( \mu \in \mathbb{R} \):

\[ \Delta (u''(\mu + \Delta)) \quad \text{strictly increasing in} \quad \Delta \geq 0 \quad \text{and} \quad \lim_{\Delta \to -\infty} \Delta u'(\mu + \Delta) = \infty \]  \hfill (2.51)

hold, then,

(i) the demand for risk

\[ s(w, \rho) := \arg \max_\sigma u(Rw + \rho \sigma) - v(\sigma) \]

is well defined and continuously differentiable in \((w, \rho)\),

(ii) \[ s(w, \rho) = 0 \quad \text{if and only if} \quad \rho = 0, \]  \hfill (2.52)

(iii) \[ \lim_{\rho \to -\infty} s(w, \rho) = \infty \quad \text{for every} \quad w \in \mathbb{R} \]  \hfill (2.53)

(iv) \[ \frac{\partial}{\partial w} s(w, \rho) = \frac{Ru''(Rw + \rho s(w, \rho))}{v''(s(w, \rho)) - \rho^2 u''(Rw + \rho s(w, \rho))}, \]

i. e. risk is an inferior good, whenever \( u \) is strictly concave,

(v) \[ \frac{\partial}{\partial \rho} s(w, \rho) = \frac{u''(Rw + \rho s(w, \rho)) + \rho s(w, \rho) u''(Rw + \rho s(w, \rho))}{v''(s(w, \rho)) - \rho^2 u''(Rw + \rho s(w, \rho))} > 0 \]  \hfill (2.54)

i. e. the demand for risk is increasing in the shadow return \( \rho \).

These relatively strong requirements for separable \( \mu - \sigma \) preferences are somewhat surprising, but they represent the subtle balance between wealth effects and substitution effects to induce monotonic demand. As the proof below shows, they are essentially necessary. At the same time, they integrate the quasi linear utility and the strict concavity into one uniform set of assumptions. Regarding wealth effects they imply that the associated expansion paths are never upward sloping. The two situations of quasi linearity \((u'' = 0)\) and \((v'' = 0)\) are boundary cases. Observe that the theorem does not cover the case with piece wise quasi linear functions \( u \), since these violate the assumptions (2.51). In such cases, the demand for risk will not be monotonic in the shadow return (see Figure 3.5).

It is also somewhat unexpected that the sign of the price effect on the demand for risk is exclusively determined by second order properties of the function \( u \). Standard economic
intuition would suggest that the function \( v \) also captures part of an agent’s attitude toward risk, which seems counterintuitive to the result that \( v \) has no influence on the sign of the derivative of demand for risk\(^{12}\). The assumption (2.51) implies that \( u' \) is convex but that its steepness is bounded.

**Proof:**
The function \( H(\rho, \sigma) := u(Rw + \rho\sigma) - v(\sigma) \) is strictly concave in \( \sigma \) for every \( \rho \geq 0 \). Since
\[
\frac{\partial H}{\partial \sigma}(\rho, \sigma) = \rho u'(Rw + \rho\sigma) - v'(\sigma),
\]
once you find
\[
\frac{\partial H}{\partial \sigma}(0, \sigma) = 0 \quad \text{if and only if} \quad \sigma = 0,
\]
since \( v'(0) = 0 \) which proves (2.52). If \( \rho > 0 \), then
\[
\frac{\partial H}{\partial \sigma}(\rho, \sigma) = 0
\]
has exactly one positive solution. Hence, \( s(w, \rho) \) is a function. The strict convexity of the function \( v \) implies assumption (2.44). Hence, \( s(w, \rho) \) is differentiable.

Property (2.53) follows from the first order conditions
\[
0 = \frac{\partial H}{\partial \sigma}(\rho, s(w, \rho)) = u'(Rw + \rho s(w, \rho)) \left( \rho - \frac{v'(s(w, \rho))}{u'(Rw + \rho s(w, \rho))} \right),
\]
and from (2.51).

Inferiority of risk (2.54) follows directly from strict concavity and an application of the implicit function theorem, while monotonicity of demand (2.55) follows from condition (2.51). QED.

Notice that no restriction for the wealth to be positive is required. If this is case one has the following corollary.

**Corollary 2.1**
Consider the separable case of theorem (??) and assume conditions (2.2) and (3.14). If for all \( \mu \geq 0 \)
\[
\mu u'(\mu) \quad \text{is strictly increasing} \quad \quad (2.56)
\]
and
\[
lim_{\mu \to \infty} \mu u'(\mu) = \infty \quad \quad (2.57)
\]
hold, then, for given non negative wealth \( w \geq 0 \), the demand for risk \( s(w, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) is monotonically increasing and bijective in \( \rho \) on \( \mathbb{R}_+ \).

\(^{12}\)This confirms essentially for the separable case that measures of risk aversion which use properties of the function \( u \) alone are sufficient in capturing all relevant effects. Whether this is true in general still is an open question. (compare Hens, Laitenberger & Löffler (2002)).
Proof:
Equation (2.55) implies
\[
\frac{\partial}{\partial \rho} s(w, \rho) = \frac{u'(Rw + \rho s(w, \rho))}{v''(s(w, \rho)) - \rho^2 u''(Rw + \rho s(w, \rho))} \left[ 1 + \frac{\rho s(w, \rho) u''(Rw + \rho s(w, \rho))}{u'(Rw + \rho s(w, \rho))} \right] - \frac{1}{\rho} \frac{u' s(w, \rho)}{v(s(w, \rho))} \left[ 1 + \frac{(Rw + \rho s(w, \rho)) u''(Rw + \rho s(w, \rho))}{u'(Rw + \rho s(w, \rho))} \right] \tag{2.58}
\]
\[
= \frac{u'(Rw + \rho s(w, \rho))}{v''(s(w, \rho)) - \rho^2 u''(Rw + \rho s(w, \rho))} \left[ 1 + \frac{E u''(Rw + \rho s(w, \rho))}{u'(Rw + \rho s(w, \rho))} \right] \tag{2.59}
\]
since the assumption (2.56) is equivalent to
\[
E u''(\mu) := \frac{\mu u''(\mu)}{u'(\mu)} > -1 \quad \text{for all } \mu \geq 0, \tag{2.60}
\]
QED.

The assumption 2.60 on the elasticity of the derivative of \( u \) was used also by Dana (1999).
Notice, that the elasticity assumption alone does not guarantee the surjectivity of risk demand\(^\text{13}\).

The method of proof also reveals that a joint condition like (2.51) is required to guarantee surjectivity of the demand for risk (2.53). It is not sufficient to assume that the marginal utility of return tends to zero with infinite shadow return to risk, since it cannot be excluded that the demand for risk remains bounded. Thus, the condition on the strong marginal disutility of risk as in (2.51) is needed to generate unbounded demand for risk.

In summary, the curvature of \( u \) alone determines the sign of the price effect, and thus whether the associated offer curve is monotonic. However, the two situations of a linear and of a strictly concave utility function \( u \) seem to require a different treatment when surjectivity is required, since the linear case is not a direct boundary situation of the strictly concave situation.

The demand for risk is always monotonically increasing in the shadow return and tending to infinity if risk is a normal good even for non separable preferences. This includes the quasi linear case of a (weakly) concave utility to expected return. However, if the utility of expected return is not linear (strictly concave or piece wise linear) strong additional properties on the curvature of \( u \) are required, since then risk becomes an inferior good.

Figures 3.3, 3.2, and 3.4 display three possible offer curves violating theorem (2.2). As in standard demand theory, their curvature is restricted only by quasi concavity, implying the usual star shaped form and allowing for inferiority of both goods as well as Giffen effects. Note, that an offer curve can be backward bending for a large class of concave utility function.

3 Asset Market Equilibrium

Let each consumers \( a \in A \) now be characterized by mean–variance preferences \( U^a \), initial wealth \( w^a \), and beliefs \((q^a, V^a)\). Let \( 0 \neq \bar{x} \in \mathbb{R}^K \) denote aggregate asset supply. If beliefs
\[\text{for example: } u(\mu) := \ln(r + \mu), r > 0 \text{ implies (2.56) but not (2.57)}\]
of consumers are not the same, then one has to distinguish for each \( a \in A \) a pair of first and second moment beliefs \((q^a, V^a)_{a \in A}\).

Following standard equilibrium terminology an asset allocation \((x^a)_{a \in A}\) is said to be feasible if \( \sum_a x^a = \bar{x} \). Therefore, for given beliefs, an asset market equilibrium is obtained by a vector of equilibrium asset prices such that

\[
\sum_{a \in A} \varphi^a(w^a, q^a - Rp) = \bar{x}.
\] (3.1)

Any vector of equilibrium asset prices induces a feasible asset allocation \((x^a)_{a \in A}\). Since individual demand is a function of subjective asset premia \( q^a - Rp \), heterogeneous first moment beliefs enter explicitly as an argument (as an additive shift of prices) in each demand function while second moment beliefs \((V^a)_{a \in A}\) determine the functional form parametrically. This fact is typically suppressed in the standard analysis.

In the context of mean–variance preferences the risk of a portfolio \( x \) is measured by its (subjective) variance or standard deviation \( \sigma(x) := \sqrt{\langle x, Vx \rangle} \). Therefore, given individual second moment beliefs \((V^a)_{a \in A}\), an asset allocation \((x^a)_{a \in A}\) induces a risk allocation \((\sigma^a)_{a \in A}\) by

\[
\sigma^a := \langle x^a, V^ax^a \rangle^{\frac{1}{2}} \quad a \in A.
\]

**Definition 3.1**

Given individual second moment beliefs \((V^a)_{a \in A}\) and an aggregate stock of assets \( \bar{x} \neq 0 \), an asset allocation \((x^a)_{a \in A}\) is said to induce an efficient risk allocation \((\sigma^a)_{a \in A}\), if there is no other feasible asset allocation \( y^a, a \in A \) such that

\[
\sqrt{\langle y^a, V^ay^a \rangle} \leq \sqrt{\sigma^a} \text{ for all } a \in A
\] (3.2)

with at least one strict inequality.

It is immediate that the set of feasible risk allocations

\[
S := \left\{ (\sigma^a) \in \mathbb{R}_+^{|A|} \mid \sigma^a := \langle x^a, V^ax^a \rangle^{\frac{1}{2}}, \quad a \in A, \sum_{a \in A} x^a = \bar{x} \right\}
\] (3.3)

is closed, bounded below, and convex. Moreover, if second moment beliefs are identical, the efficient boundary \( S_{eff} \) is given by the simplex

\[
S_{eff} = \left\{ (\sigma^a) \in \mathbb{R}_+^{|A|} \mid \sum_{a \in A} \sigma^a = \langle \bar{x}, V\bar{x} \rangle^{\frac{1}{2}} \right\}
\] (3.4)

Figure 3.1 shows the set of feasible allocations in an Edgeworth box representation for two agents and two assets displaying the contours of the two variances \( \sigma(x^a) \) and \( \sigma(\bar{x} - x^a) \). The ”contract curve” connecting the corners \( a \) and \( b \) is the (the projection) of the set of efficient risk allocations in the space for agent \( a \)' assets. If \( V^a = V^b \) the contract curve is a straight line.
3.1 Existence and Uniqueness of Equilibrium

If first moment expectations are identical, any equilibrium asset price vector induces the same equilibrium asset premium, the equilibrium condition can be rewritten as $\pi \in \mathbb{R}^K$ if

$$\sum_{a \in A} \varphi_a(w^a, \pi) = \bar{x}. \tag{3.5}$$

In such a case one obtains the following lemma.

**Lemma 3.1**

*If first moment beliefs are identical, any asset market equilibrium induces an efficient risk allocation.*

### 3.1 Existence and Uniqueness of Equilibrium

Continuing with the assumption of common (identical) expectations satisfying (2.1) for all consumers$^{14}$, an asset market equilibrium is achieved for a vector of subjective premia $\pi$, such that

$$\sum_{a \in A} \varphi_a(w^a, \pi) = \bar{x}. \tag{3.6}$$

Exploiting the features of the factorization lemma (2.2), one finds that $\pi$ is an equilibrium premium if and only if the condition

$$\left( \sum_{a \in A} s^a(w^a, \langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}) \right) \frac{1}{\langle \pi, V^{-1} \pi \rangle^{\frac{1}{2}}} V^{-1} \pi = \bar{x} \tag{3.7}$$

---

$^{14}$Existence of equilibria with heterogeneous beliefs will be discussed in a later section.
holds. Let \( \rho := (\pi, V^{-1}\pi)^{\frac{1}{2}} \) and define aggregate demand for risk as
\[
s((w^a)_{a \in A}, \rho) := \sum_{a \in A} s^a(w^a, \rho)
\]

**Lemma 3.2**

If aggregate demand for risk \( s((w^a)_{a \in A}, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is continuous and surjective in the shadow return \( \rho \), there exists an asset market equilibrium for every aggregate asset supply \( \bar{x} \neq 0 \) and arbitrary homogeneous beliefs \((q, V)\) satisfying (2.1). If, in addition, demand for risk is strictly monotonic in \( \rho \), then asset equilibrium is unique with an equilibrium premium given by
\[
\pi = \frac{1}{\sqrt{\langle \bar{x}, V\bar{x} \rangle}} s^{-1}((w^a)_{a \in A}, \sqrt{\langle \bar{x}, V\bar{x} \rangle}) V\bar{x} \tag{3.8}
\]

and associated equilibrium asset prices
\[
p = \frac{1}{R} \left( q - \frac{1}{\sqrt{\langle \bar{x}, V\bar{x} \rangle}} s^{-1}((w^a)_{a \in A}, \sqrt{\langle \bar{x}, V\bar{x} \rangle}) V\bar{x} \right) \tag{3.9}
\]

where \( s^{-1}((w^a)_{a \in A}, \cdot) \) is the inverse of the aggregate demand function for risk with respect to \( \rho \).

Notice that the existence of an equilibrium in the markets for the \( K \) assets is reduced to finding a zero of the one dimensional mapping of excess demand for risk\(^{15}\).

**Proof:**

From the asset market equilibrium equation (3.7)
\[
\left( \sum_{a \in A} s^a(w^a, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}) \right) \frac{1}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi = \bar{x}
\]

it follows that any equilibrium premium \( \pi \) must be colinear to \( V\bar{x} \). Choose \( \lambda > 0 \) arbitrary and define \( \pi := \lambda V\bar{x} \neq 0 \). Then,
\[
\bar{x} = \left( \sum_{a \in A} s^a(w^a, \langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}) \right) \frac{1}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi \tag{3.10}
\]

\[
\iff \bar{x} = \frac{s((w^a)_{a \in A}, \lambda \langle \bar{x}, V\bar{x} \rangle^{\frac{1}{2}})}{\langle \bar{x}, V\bar{x} \rangle^{\frac{1}{2}}} \bar{x} \tag{3.11}
\]

\[
\iff s((w^a)_{a \in A}, \lambda \sqrt{\langle \bar{x}, V\bar{x} \rangle}) = \sqrt{\langle \bar{x}, V\bar{x} \rangle} \tag{3.12}
\]

In other words, \( \pi := \lambda V\bar{x} \) is an equilibrium premium if and only if aggregate demand for risk is equal to aggregate risk for its induced shadow return \( \lambda \sqrt{\langle \bar{x}, V\bar{x} \rangle} \). Therefore,
continuity and surjectivity of the left hand side of (3.12) in \( \lambda \) implies existence of an asset market equilibrium. Strong monotonicity implies the unique solution

\[
\lambda^* = \frac{1}{\sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle}} s^{-1}\left((w^a)_{a \in A}, \sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle}\right)
\]

Thus, \( \rho^* = \lambda^* \sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle} \) is the equilibrium shadow return to risk, i.e.

\[
\sum_{a \in A} s^a(w^a, \lambda^* \sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle}) = \sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle},
\]

implying that the sum of individual risk is equal total risk. This yields the equations for the unique equilibrium premium (3.8) and for the equilibrium asset price vector (3.9).

QED.

We close this section with two theorems of existence and uniqueness by stating two sets of economic assumptions on preferences, beliefs, and wealth to guarantee a unique asset market equilibrium.

**Theorem 3.1**

Let beliefs of consumers be identical and satisfy (2.1). Assume that preferences of all consumers \( a \in A \) fulfill assumptions (2.3), and the Inada conditions (2.41).

If risk is a normal good for all consumers and at least one consumer has quasi linear preferences in \( \mu \), there exists a unique asset market equilibrium for any expectations \((q, V)\) and any aggregate supply of assets \( \mathbf{x} \neq 0 \).

**Proof:**

In view of lemma (3.2) it is sufficient to show that aggregate demand for risk is continuous, strictly monotonic and surjective in \( \rho \). Assumptions (2.1), (2.3), and the Inada conditions (2.41) imply that individual demand for risk exists and is a continuous function for all \( \rho > 0 \) with

\[
\lim_{\rho \to 0} s^a(w^a, \rho) = 0.
\]

Normality of risk implies that the demand for risk is increasing in \( \rho \). Moreover, the Inada conditions and quasi linearity for some agent \( a \) imply that

\[
\lim_{\rho \to \infty} s^a(w^a, \rho) = \infty.
\]

Hence, aggregate demand is surjective and there exists a unique shadow return \( \rho^* \) with

\[
\rho^* = s^{-1}\left((w^a)_{a \in A}, \sqrt{\langle \mathbf{x}, V \mathbf{x} \rangle}\right)
\]

QED.

The assumption that risk is a normal good for all consumers seems fairly restrictive, since there is a large set of reasonable preferences for which risk is an inferior good (as in examples (2.32) and (2.29)). Therefore, it is desirable to obtain existence and uniqueness in such cases as well. The following theorem provides a positive answer for the subclass of separable preferences.
**Theorem 3.2**

Assume that beliefs of consumers are identical satisfying (2.1) and that preferences satisfy assumption (2.3). If, for all consumers \( a \in A \), one has

\[
(v^a)'(0) = 0, \quad \lim_{\sigma \to \infty} (v^a)'(\sigma) = \infty, \quad \text{and} \quad (v^a)''(\sigma) > 0 \quad \text{for all} \quad \sigma > 0, \tag{3.14}
\]

\[
\Delta (u^a)'(\mu + \Delta) \quad \text{strictly increasing in} \quad \Delta \geq 0 \quad \text{for all} \quad \mu \in \mathbb{R}, \tag{3.15}
\]

and for at least one consumer

\[
\lim_{\Delta \to -\infty} \Delta (u^a)'(\mu + \Delta) = \infty \tag{3.16}
\]

Then there exists a unique asset market equilibrium for any expectations \((q, V)\) and any aggregate supply of assets \( \bar{x} \neq 0 \).

**Proof:**

Assumptions (2.1) and (2.3) for all consumers \( a \in A \) imply that aggregate demand for risk is differentiable. It follows from (2.55) that the demand for risk is an increasing function for every agent \( a \in A \). In addition, (3.16) for at least one agent implies that \( \lim_{\rho \to -\infty} s^a(w^a, \rho) = \infty \) which yields this property for the aggregate demand for risk. Therefore, aggregate demand for risk satisfies the conditions of lemma (3.2). QED.

Observe that both of the preceding theorems include in their assumptions the case of quasi linear preferences in \( \mu \) for all agents. However, for preferences with strict concavity in \( \mu \), the two sets of assumptions are mutually exclusive, since risk is an inferior good under separability. The condition (3.16) is crucial in guaranteeing existence, while the monotonicity condition (3.15) assures uniqueness only. Example 3 provides a case where the monotonicity condition is satisfied but demand for risk is bounded. For such a situation it is easy to construct examples with no equilibrium and beliefs satisfying assumptions (2.1).\(^{16}\)

The factorization lemma (2.2) implies that individual asset demand is given by

\[
\varphi^a(w^a, \bar{x}) = \frac{s^a(w^a, \bar{\rho})}{\sqrt{\langle \bar{x}, V\bar{x} \rangle \bar{x}}} \bar{x} = \frac{s^a(w^a, \bar{\rho})}{\bar{\sigma}} \bar{x}
\]

which is a positive multiple of the market portfolio. Thus, in equilibrium agents hold the same mix of assets as the market portfolio and they share aggregate risk proportionally, confirming the mutual fund result of the CAPM literature.

Finally, asset prices are given by

\[
p = \frac{1}{R} \left[ q - \frac{\bar{\rho}}{\langle \bar{x}, V\bar{x} \rangle ^{\frac{1}{2}}} \right] V\bar{x} \tag{3.17}
\]

such that \( s((w^a)_{a \in A}, \bar{\rho}) = \langle \bar{x}, V\bar{x} \rangle ^{\frac{1}{2}} \). Thus, equilibrium asset prices are an affine map in expected prices plus an additive constant which depends on preferences, second moment beliefs, and aggregate asset supply. This corresponds to a form of the 'beta pricing rule', which can be transformed directly into the usual affine relation ship between returns. Observe that equilibrium prices are bounded above by the discounted subjective first
3.1 Existence and Uniqueness of Equilibrium

Figure 3.2: No equilibrium $\hat{x}$

Figure 3.3: Two equilibria
existence and uniqueness of equilibrium

moment expectations if $V\bar{x}$ is non negative, and prices are positive only if expectations are sufficiently optimistic. Figures (3.2), (3.3), and (3.4) provide geometric insight into the existence and uniqueness issue. They display the aggregate offer curve for three different situations for which existence or uniqueness fails. In the case of piece wise quasi linear preferences it is apparent that the demand for risk cannot be differentiable and monotonically increasing for all wealth levels. There can be multiple sections with deceasing demand for risk. This implies the possibility of an arbitrary finite number of equilibria, a situation depicted in Figure 3.5.

\[\begin{aligned}
\sum R_w^a & \\
\sqrt{\langle \bar{x}, V \bar{x} \rangle} & \\
\end{aligned}\]

Figure 3.4: Three equilibria

\[\begin{aligned}
\mu & \\
\sigma & \\
\end{aligned}\]

\[\begin{aligned}
\sum R_w^a & \\
\sqrt{\langle \bar{x}, V \bar{x} \rangle} & \\
\end{aligned}\]

\[\begin{aligned}
\mu & \\
\sigma & \\
\end{aligned}\]

16 Dana (1999) uses only (3.15) for positive $\mu$ in her proof of existence.
3.2 The equilibrium set and determinacy

In the case of uniqueness we can now summarize the results as follows providing a complete characterization of asset prices under general and homogeneous beliefs for arbitrary preferences, initial wealth, and aggregate stock of assets.

Let \((q, V)\) and \(\bar{x} \neq 0\) be given and \(\bar{\sigma} := \langle \bar{x}, V \bar{x} \rangle^{\frac{1}{2}}\) denote the market risk. Define \(\bar{\rho} > 0\) to be the unique market clearing shadow return; i.e.

\[
    s(((w^a)_{a \in A}, \bar{\rho}) = \langle \bar{x}, V \bar{x} \rangle^{\frac{1}{2}}. \tag{3.18}
\]

If the demand for risk is differentiable and regular, standard results from the theory of regular economies ((c.f. Mas-Colell 1985)) provide the parallel results for the case here. For a smooth regular aggregate demand function for risk, equilibria are determinate and generically finite, implying the typical structure of the equilibrium manifold for given preferences and second moment beliefs

\[
    W_\rho := \{(w, \sigma, \rho) \in \mathbb{R}^{|A|} \times \mathbb{R}^2_+ \mid s(((w^a)_{a \in A}, \rho) = \sigma)\}. \tag{3.19}
\]

Figures (3.6), (3.7), and (3.8) show the three essential cases which may occur.

Since the asset price equation (3.17) is a smooth mapping, the equilibrium manifold for asset prices

\[
    W := \{(w, q, V, \bar{x}, p) \mid s(w, \bar{\rho}) = \langle \bar{x}, V \bar{x} \rangle^{\frac{1}{2}}, \quad \bar{\rho} > 0 \}
\]

\[
    p = \frac{1}{R} \left[ q - \frac{\bar{\rho}}{\langle \bar{x}, V \bar{x} \rangle^{\frac{1}{2}}} V \bar{x} \right]. \tag{3.20}
\]
3.2 The equilibrium set and determinacy

Figure 3.6: Unique equilibrium $\hat{x}$

Figure 3.7: Global equilibrium manifold $\hat{x}$
3.2 The equilibrium set and determinacy

Figure 3.8: Bounded equilibrium manifold $\mathcal{W}_\rho$ with $w := (w^a)_{a \in A}$ and

$$(w, q, V, \bar{x}, p) \in \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{K} \times \mathbb{R}^{K^2} \times \mathbb{R}^{K}_+ \times \mathbb{R}^{K}$$

is well behaved in general. Thus, the following proposition is an immediate consequence of standard results from smooth equilibrium analysis (see for example Mas-Colell (1985)).

**Proposition 3.1**

Let beliefs satisfy (2.1) and assume that aggregate asset demand $s(w, \bar{\rho})$ is regular. Then,

$$\mathcal{W} \subset \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{K} \times \mathbb{R}^{K^2} \times \mathbb{R}^{K}_+ \times \mathbb{R}^{K}$$

is a smooth manifold. Thus, the set of equilibrium asset prices is finite.

**Proof:**
The equilibrium manifold $\mathcal{W}$ is the graph of the smooth price map (3.17) restricted to the smooth manifold (3.19). QED.

Summarizing the results so far, one observes that existence and uniqueness of equilibrium are guaranteed under a set of intuitive economic assumptions on preferences and beliefs for the mean variance case. The fact that beliefs are homogeneous (identical) reduces the existence issue to finding a zero of a one dimensional excess demand function for risk, formally similar to the reduction used by Dana (1999), but conceptually quite distinct. At the same time, such equilibria are mean variance efficient given expectations. Thus,
in equilibrium, asset bundles are all positive multiples of the aggregate market portfolio. As a consequence, the equilibrium shadow return to risk $\hat{\rho}$ is the same for all consumers and equal to the subjective Sharpe ratio (i.e. the ratio of expected return to standard deviation). Thus, individual risk taking induces an efficient (ex ante) risk sharing.

## 4 Equilibria with asset endowments

Consider the capital asset pricing model as a pure exchange model with private ownership of asset endowments. In this situation total wealth of agents is endogenous and price dependent. It is straightforward to extend the above model to the case with individual asset endowments and endogenous wealth.

Let $(e^a, \bar{x}^a) \in \mathbb{R}^{K+1}_+$, $a \in A$, denote agent $a$’s endowment of the numeraire commodity $e^a \in \mathbb{R}_+$ and of the risky assets $\bar{x}^a \in \mathbb{R}^K_+$, with total asset endowments $\bar{x} = \sum_A \bar{x}^a$. Then, the list $\{(U^a, (e^a, \bar{x}^a))\}_{a \in A}$ is an exchange economy in the usual sense.

### Definition 4.1

A mean–variance asset exchange economy with private ownership $\mathcal{E}$ is a list

$$
\mathcal{E} := \{(U^a, (e^a, \bar{x}^a), (q^a, V^a))\}_{a \in A}
$$

such that for all $a \in A$:

(i) $(e^a, \bar{x}^a) \in \mathbb{R}^{K+1}_+$

(ii) $(e^a, \bar{x}^a) \in \mathbb{R}^{K+1}_+$

(iii) $U^a$ satisfies assumption (2.2)

For any asset price vector $p \in \mathbb{R}^K$, the relationship between the previous model and the private ownership situation is the usual one. For every agent $a \in A$, define the wealth function

$$
\mathcal{W}^a(p) := e^a + \langle p, \bar{x}^a \rangle.
$$

and, for given beliefs $(q^a, V^a)$, consider the equilibrium asset price equation (3.17) as a general price law of $P : \mathbb{R}_+ \rightarrow \mathbb{R}^K_+$ defined by

$$
p = P(\rho) := \frac{1}{R} \left[ q - \frac{\rho}{\langle \bar{x}, V \bar{x} \rangle^{1/2}} V \bar{x} \right].
$$

### 4.1 Homogeneous Beliefs

### Definition 4.2

Given homogeneous beliefs $(q, V)$, an asset price vector $\bar{p} \in \mathbb{R}^K$ is an equilibrium price vector if there exists a shadow return $\bar{\rho} \geq 0$ and a wealth distribution $\bar{w} := (\bar{w}^a)_{a \in A} \in \mathbb{R}^{|A|}$,
4.1 Homogeneous Beliefs

such that

\[ \bar{x} = \sum_{a \in A} \bar{x}^a \] (4.4)

\[ \langle \bar{x}, V \bar{x} \rangle^\frac{1}{2} = \sum_{a \in A} s^a(\bar{\omega}^a, \bar{\rho}) \] (4.5)

\[ \bar{p} = \frac{1}{R} \left[ q - \frac{\bar{\rho}}{\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}} V \bar{x} \right] \] (4.6)

\[ \bar{\rho} = \langle (q - R\bar{p}), V^{-1}(q - R\bar{p}) \rangle^\frac{1}{2} \] (4.7)

\[ \bar{\omega}^a = \mathcal{W}^a(\bar{\rho}) = e^a + \langle \bar{\rho}, \bar{x}^a \rangle \quad a \in A \] (4.8)

In other words an asset market equilibrium is a fixed point of the four mappings given by the right hand sides of equations (4.5)–(4.8). Therefore existence and uniqueness can now be proved exploiting the properties derived in the previous sections. The two results are directly parallel of the two previous theorems 3.1 and 3.2.

**Theorem 4.1**

Let individual endowments \((e^a, \bar{x}^a) \geq 0\) with \(\bar{x} = \sum_{a \in A} \bar{x}^a \neq 0\), \(\sum_{a \in A} e^a > 0\). Assume that preferences of all consumers \(a \in A\) fulfill assumptions (2.3), and the Inada conditions (2.41).

If risk is a normal good for all consumers and at least one consumer has quasi linear preferences in \(\mu\), there exists a unique equilibrium asset price \(\bar{p} \in \mathbb{R}^K\) for the private ownership economy for all beliefs satisfying (2.1) such that \(q >> 0\) and \(V \bar{x} \gg 0\).

The second theorem which allows for risk inferiority parallels that of Theorem (3.2).

**Theorem 4.2**

Let individual endowments \((e^a, \bar{x}^a) \geq 0\) with \(\bar{x} = \sum_{a \in A} \bar{x}^a \neq 0\), \(\sum_{a \in A} e^a > 0\). If, for all consumers \(a \in A\), preferences are separable with 2.3 such that:

\[ (v^a)'(0) = 0, \quad \lim_{\sigma \to \infty} (v^a)'(\sigma) = \infty, \quad \text{and} \quad (v^a)''(\sigma) > 0 \quad \text{for all} \quad \sigma > 0, \] (4.9)

\[ \Delta (u^a)'(\mu + \Delta) \quad \text{strictly increasing in} \quad \Delta \geq 0 \quad \text{for all} \quad \mu \in \mathbb{R}, \] (4.10)

and for at least one consumer

\[ \lim_{\Delta \to -\infty} \Delta (u^a)'(\mu + \Delta) = \infty \] (4.11)

there exists a unique equilibrium asset price \(\bar{p}\) for the private ownership economy for all beliefs satisfying (2.1) such that \(q >> 0\) and \(V \bar{x} \gg 0\).

Notice that the conditions (4.9)–(4.11) are the same as (3.14)–(3.16). We only give the proof of the extension for the case of Theorem (3.2).
4.1 Homogeneous Beliefs

Proof:
Theorem (3.2) implies the existence of a continuous function \( r : \mathbb{R}^{|A|} \to \mathbb{R}_+ \) defining the unique equilibrium shadow return \( \rho = r(w) \) for every initial wealth \( w = (w^a)_{a \in A} \), i.e. \( r \) is the solution function for equation (4.7).

\[
\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2} = s(w, r(w)) = \sum_a s^a(w^a, r(w)).
\]

Consider the mapping

\[
H := \left\{ \begin{array}{c}
\mathbb{R}^{|A|} \\
w
\end{array} \to \mathbb{R}^{|A|} \right.
\]

defined by

\[
H(w) = (H^a(w))_{a \in A} \quad H^a(w) := (W^a o P o r)(w)
\]

(4.12)

\( H \) is continuous and each \( H^a(w) \) is bounded above by

\[
\bar{w} := \sum e^a + \langle \bar{x}, \frac{1}{R} q \rangle
\]

and below by

\[
w := \min_{a \in A} \left( e^a + \langle \bar{x}, \frac{1}{R} q \rangle - \frac{r(w)}{\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}} V \bar{x} \right)
\]

Hence, restricting \( H \) to the compact convex set

\[
W := \{ w \in \mathbb{R}^{|A|} | w \leq w^a \leq \bar{w}, a \in A \}
\]

implies \( H : W \to W \). Since \( H \) is continuous, there exists a fixed point \( w^* \in W \). This induces an equilibrium shadow return \( p^* = r(w^*) \) and equilibrium asset prices

\[
p^* = \frac{1}{R} \left[ q - \frac{\rho^*}{\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}} V \bar{x} \right]
\]

(4.13)

To prove uniqueness, assume to the contrary that there exists two fixed points \( w^1 \neq w^2 \). If \( \rho^1 = r(w^1) = r(w^2) = \rho^2 \), there exist a unique asset price from equation (4.13). Suppose \( \rho^1 > \rho^2 > 0 \). This yields

\[
p^1 = \frac{1}{R} \left[ q - \frac{\rho^1}{\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}} V \bar{x} \right] < p^2 = \frac{1}{R} \left[ q - \frac{\rho^2}{\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}} V \bar{x} \right],
\]

implying \( w^2 > w^1 \) since endowments are positive. Since preferences are assumed to be separable and strictly concave, risk is an inferior good. The assumption (??) implies that the demand for risk increases in the shadow return \( \rho \). Therefore, one finds that

\[
s^a(w^1, \rho^1) > s^a(w^2, \rho^1) > s^a(w^2, \rho^2)
\]

so that

\[
\langle \bar{x}, V \bar{x} \rangle^\frac{1}{2} = \sum_A s^a(w^2, \rho^2) > \sum_a s^a(w^1, \rho^1) = \langle \bar{x}, V \bar{x} \rangle^\frac{1}{2}
\]

which is a contradiction.

QED.

The following example indicates that an assumption on the limiting behavior such as (2.56) cannot be dispensed with if existence is to be guaranteed.
Example

Consider an exchange economy with two agents $A = \{a, b\}$

\[ 0 < r < \langle \bar{x}, V\bar{x} \rangle \text{ and } q \gg 0 \]

. There exists $H \subset \{1, \ldots, K\}$ such that

\begin{align*}
(V\bar{x})_k &< 0 & k \in H, & (V\bar{x})_k > 0 & k \notin H, \\
(\bar{x}^a)_k &> 0 & k \in H, & (\bar{x}^a)_k = 0 & k \notin H, \\
(\bar{x}^b)_k &= 0 & k \in H, & (\bar{x}^b)_k > 0 & k \notin H, \\
U^a(\mu, \sigma)) &= \ln(r + \mu) - \sigma^2 & e^a > 0, & (4.14) \\
U^b(\mu, \sigma)) &= -\sigma^2 & e^b = 0. & (4.15)
\end{align*}

It is apparent that consumer $b$ will supply $(\bar{x})^b$ at any price with zero demand for risk. Notice that, because of the negative correlation for commodities $k \in H$, equilibrium prices for these commodities must be positive, since

\[ p = \frac{1}{R} \left( q - \frac{\lambda}{\langle \bar{x}, V\bar{x} \rangle^2} V\bar{x} \right). \]

Therefore, $a$’s wealth will be larger than $e^a > 0$ implying that his demand for risk will be less than $r$. Hence, there is excess supply of risk for all asset prices.

Finally, Figure 4.1 portrays the situation of multiple equilibria with two agents $A = \{a, b\}$ where $b$ has quasi linear preferences and $a$ has a piece wise linear utility in the return $\mu$.

5 Conclusion

Under a general non redundancy assumption on expectations, the CAPM with mean–variance preferences induces demand behavior explicitly derivable from intuitive assumptions using standard concepts like normality and the Slutsky decomposition. For homogeneous expectations globally invertible demand for commodities $k \in H$, equilibrium prices for these commodities must be positive, since

The results of the paper have shown for the CAPM model, that existence and uniqueness of equilibria are essentially determined by those properties of risk preferences in mean in standard deviation which guarantee globally invertible demand functions for risk, when agents hold arbitrary but homogeneous beliefs. These features have an immediate impact on the equilibrium set, inducing typically smooth manifolds when preferences are smooth.
Figure 4.1: Multiple Equilibria with two agents
References


